Dynamic Mechanism Design Tutorial

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Recursive Mechanisms with Transfers

Dynamic Games without Enforcement and without Transfers
A Simple Example

1a. Seller learns $\theta_S$
1b. Buyer buys $x_1$ from Seller
2a. Buyer learns $\theta_B$
2b. Buyer buys $x_2$ from Seller

- Buyer’s total value: $x_1 + \theta_B x_2$
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- Seller’s cost $c(x_t, \theta_S) = \frac{1}{2} x_t^2 / \theta_S$ in each period $t = 1, 2$. 
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- Seller’s cost $c(x_t, \theta_S) = \frac{1}{2} x_t^2 / \theta_S$ in each period $t = 1, 2$.
- Efficient plan: $\chi_1(\theta_S) = \theta_S$, $\chi_2(\theta_S, \theta_B) = \theta_S \theta_B$
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  - Note: B infers $\theta_S$ from $\chi_1(\theta_S)$
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- Team Transfers (not BB):
  \[
  \gamma_S(\hat{\theta}_B, \hat{\theta}_S) = \chi_1(\hat{\theta}_S) + \hat{\theta}_B \cdot \chi_2(\hat{\theta}_S, \hat{\theta}_B),
  \gamma_B(\hat{\theta}_B, \hat{\theta}_S) = -c(\chi_2(\hat{\theta}_S, \hat{\theta}_B), \hat{\theta}_S).
  \]
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- Efficient plan: $\chi_1(\theta_S) = \theta_S, \chi_2(\theta_S, \theta_B) = \theta_S \theta_B$
  - Note: B infers $\theta_S$ from $\chi_1(\theta_S)$
- Static AGV (“Expected Externality”)—note beliefs are CK:
  
  $\gamma_S(\hat{\theta}_S) = \chi_1(\hat{\theta}_S) + \mathbb{E}_{\tilde{\theta}_B} [\tilde{\theta}_B \cdot \chi_2 (\hat{\theta}_S, \tilde{\theta}_B)]$,
  
  $\gamma_B (\hat{\theta}_B) = -\mathbb{E}_{\tilde{\theta}_S} [c (\chi_2(\tilde{\theta}_S, \hat{\theta}_B), \tilde{\theta}_S)]$.
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  \gamma_B (\hat{\theta}_B) = -\mathbb{E}_{\tilde{\theta}_S} [c (\chi_2(\tilde{\theta}_S, \tilde{\theta}_B), \tilde{\theta}_S)].
  \]

- Total payment from B to S:

  \[
  \psi_S(\theta_B, \theta_S) = -\psi_B(\theta_B, \theta_S) = \gamma_S(\theta_S) - \gamma_B(\theta_B)
  \]
Instead of $E_{\theta_S}$, calculate $\gamma_B$ using S’s reported $\hat{\theta}_S$:

$$\gamma_B (\hat{\theta}_S, \hat{\theta}_B) = -c \left( \chi_2(\hat{\theta}_S, \hat{\theta}_B), \hat{\theta}_S \right)$$
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But then S, who pays $\gamma_B$, would lie to manipulate it!
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Let B’s $\gamma_B = \text{change}$ in S’s expected [CP] cost induced by B’s report:

$$\gamma_B (\hat{\theta}_S, \hat{\theta}_B) = -c \left( \chi_2 (\hat{\theta}_S, \hat{\theta}_B), \hat{\theta}_S \right) + \mathbb{E}_{\tilde{\theta}_B} \left[ c \left( \chi_2 (\hat{\theta}_S, \tilde{\theta}_B), \hat{\theta}_S \right) \right]$$
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\( \gamma_B \) lets B internalize S’s cost \( \Rightarrow \) B will not lie regardless of what \( \theta_S \) he infers
Building an IC Dynamic Mechanism

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- $\mathbb{E}_{\tilde{\theta}_B} \gamma_B (\tilde{\theta}_B, \theta_S) \equiv 0 \Rightarrow$ having S pay $\gamma_B$ does not alter S’s incentives if B is truthful
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- Thus letting $\psi_S(\theta_B, \theta_S) = -\psi_B(\theta_B, \theta_S) = \gamma_S(\theta_S) - \gamma_B(\theta_B, \theta_S)$ yields a BIC balanced-budget mechanism
Generalizing Example: Add Another Period of Trade

- Seller type constant across repetitions, buyer type serially correlated
  1a. Seller learns $\theta_S$
  1b. Buyer buys $x_1$ from Seller
  2a. Buyer learns $\theta_{B,2}$
  2b. Buyer buys $x_2$ from Seller
  3a. Buyer learns $\theta_{B,3}$
  3b. Buyer buys $x_3$ from Seller

\[ \text{Pay buyer } \gamma_B = \text{change in } S' \text{'s expected cost induced by } B' \text{'s report in each repetition. Implies } \gamma_B \text{ incentive payment to buyer is:} \]

\[ \gamma_B, 3 = \hat{\theta}_S, \hat{\theta}_B, 3, \hat{\theta}_B, 2 \]

In period 2, buyer sees add' l effect of reporting $\hat{\theta}_B, 2$: affects beliefs

"Correction term" was there to neutralize seller' s incentive to manipulate $\gamma_B, 3$ through report of $\hat{\theta}_S$

But in period 2, this correction distorts buyer' s incentives
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- Pay buyer $\gamma_B = \text{change}$ in S’s expected cost induced by B’s report in each repetition. Implies $t = 3$ incentive payment to buyer is:

$$\gamma_{B,3} (\hat{\theta}_S, \hat{\theta}_{B,3}, \hat{\theta}_{B,2}) = -c \left( \chi_3(\hat{\theta}_S, \hat{\theta}_{B,3}, \hat{\theta}_S) \right) + \mathbb{E}_{\tilde{\theta}_{B,3}} \left[ c \left( \chi_1(\hat{\theta}_S, \tilde{\theta}_{B,3}, \hat{\theta}_S) \right) \bigg| \hat{\theta}_{B,2} \right].$$
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  + \mathbb{E}_{\tilde{\theta}_{B,3}} \left[ c \left( \chi_1 (\hat{\theta}_S, \tilde{\theta}_{B,3}, \hat{\theta}_S) \right) \big| \hat{\theta}_{B,2} \right].
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- In $t = 2$, buyer sees add’l effect of reporting $\hat{\theta}_{B,2}$: affects beliefs
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- “Correction term” was there to neutralize seller’s incentive to manipulate $\gamma_{B,3}$ through report of $\hat{\theta}_S$
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+ \mathbb{E}_{\tilde{\theta}_{B,3}} [c(\chi_1(\hat{\theta}_S, \tilde{\theta}_{B,3}), \hat{\theta}_S) | \hat{\theta}_{B,2}].
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- In $t = 2$, buyer sees add’l effect of reporting $\hat{\theta}_{B,2}$: affects beliefs
- “Correction term” was there to neutralize seller’s incentive to manipulate $\gamma_{B,3}$ through report of $\hat{\theta}_S$
- But in period 2, this correction distorts buyer’s incentives
The Model

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  - Each agent $i = 1, \ldots, N$ privately observes signal $\theta_{i,t} \in \Theta_{i,t}$
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  1. Each agent $i = 1, \ldots, N$ privately observes signal $\theta_{i,t} \in \Theta_{i,t}$
  2. Agents send simultaneous reports

Technology:

Preferences: Agent $i$'s utility $\delta < \delta < 1$ uniformly bounded
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  4. Mechanism makes public decision $x_{0,t} \in X_{0,t}$, transfers $y_{i,t} \in \mathbb{R}$ to each $i$
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- Histories: $\theta^t = (\theta_1, \ldots, \theta_t) \in \Theta^t = \prod_{\tau=1}^{t} \prod_{i} \Theta_{i,t}$; similarly $x^t \in X^t$
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- Technology: $\theta_t \sim \nu_t (\cdot | x^{t-1}, \theta^{t-1})$

- Preferences: Agent $i$'s utility

\[
\sum_{t=1}^{\infty} \delta^t \left[ u_{i,t}(x^t, \theta^t) + y_{i,t} \right]
\]
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- Preferences: Agent $i$’s utility

$$\sum_{t=1}^{\infty} \delta^t \left[ u_{i,t}(x^t, \theta^t) + y_{i,t} \right]$$

- $0 < \delta < 1$
The Model

- In each period \( t = 1, 2, \ldots \)
  1. Each agent \( i = 1, \ldots, N \) privately observes signal \( \theta_{i,t} \in \Theta_{i,t} \)
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\sum_{t=1}^{\infty} \delta^t \left[ u_{i,t}(x^t, \theta^t) + y_{i,t} \right]
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- \( 0 < \delta < 1 \)
- \( u_{i,t} \) uniformly bounded
Measurable decision plan: \( \chi_t : \Theta^t \rightarrow X_t \)
Direct Mechanisms

- Measurable decision plan: $\chi_t : \Theta^t \rightarrow X_t$
  - $\chi_{0,t}$ are prescribed public decisions

Information Disclosure: All announcements are public.

Disclosing less would preserve equilibrium as long as agents can still infer recommended private decisions.
Measurable decision plan: $\chi_t : \Theta^t \rightarrow X_t$

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- $\chi_{i,t}$ are recommended private decisions for agent $i \geq 1$
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- Transfers: $\psi_{i,t} : \Theta^t \to \mathbb{R}$; PDV $\Psi_i(\theta) = \sum_{t=0}^{\infty} \delta^t \psi_{i,t}(\theta^t)$
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  - $\chi_{i,t}$ are recommended private decisions for agent $i \geq 1$
- Decision plan induces stochastic process $\mu[\chi]$ on $\Theta$
- Transfers: $\psi_{i,t} : \Theta^t \rightarrow \mathbb{R}$; PDV $\Psi_i(\theta) = \sum_{t=0}^{\infty} \delta^t \psi_{i,t}(\theta^t)$
  - Measurable, uniformly bounded
Direct Mechanisms

- Measurable decision plan: $\chi_t : \Theta^t \rightarrow X_t$
  - $\chi_{0,t}$ are prescribed public decisions
  - $\chi_{i,t}$ are recommended private decisions for agent $i \geq 1$
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  - Budget balance: $\sum_i \psi_{i,t}(\theta) \equiv 0$
Direct Mechanisms

- Measurable decision plan: \( \chi_t : \Theta^t \rightarrow \chi_t \)
  - \( \chi_{0,t} \) are prescribed public decisions
  - \( \chi_{i,t} \) are recommended private decisions for agent \( i \geq 1 \)

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- Transfers: \( \psi_{i,t} : \Theta^t \rightarrow \mathbb{R} \); PDV \( \Psi_i(\theta) = \sum_{t=0}^{\infty} \delta^t \psi_{i,t}(\theta^t) \)
  - Measurable, uniformly bounded
  - Budget balance: \( \sum_i \psi_{i,t}(\theta) \equiv 0 \)

- Information Disclosure: All announcements are public
Measurable decision plan: $\chi_t : \Theta^t \rightarrow X_t$

- $\chi_{0,t}$ are prescribed public decisions
- $\chi_{i,t}$ are recommended private decisions for agent $i \geq 1$

Decision plan induces stochastic process $\mu[\chi]$ on $\Theta$

Transfers: $\psi_{i,t} : \Theta^t \rightarrow \mathbb{R}$; PDV $\Psi_i(\theta) = \sum_{t=0}^{\infty} \delta^t \psi_{i,t}(\theta^t)$

- Measurable, uniformly bounded
- Budget balance: $\sum_i \psi_{i,t}(\theta) \equiv 0$

Information Disclosure: All announcements are public

- Disclosing less would preserve equilibrium as long as agents can still infer recommended private decisions
Agent $i$’s strategy defines
Agent $i$’s strategy defines

- Reporting plan $\beta_{i,t}: \Theta_i^t \times \Theta_{-i}^{t-1} \rightarrow \Theta_{i,t}$
Agent $i$’s strategy defines:

- Reporting plan $\beta_{i,t} : \Theta_i^t \times \Theta_{-i}^{t-1} \rightarrow \Theta_{i,t}$
- Private action plan $\alpha_{i,t} : \Theta_i^t \times \Theta_{-i}^t \rightarrow X_{i,t}$
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Strategy also defines behavior following agent’s own deviations, but this is irrelevant for the normal form.
strategies

- Agent $i$’s strategy defines
  - Reporting plan $\beta_{i,t} : \Theta_{i}^t \times \Theta_{-i}^{t-1} \rightarrow \Theta_{i,t}$
  - Private action plan $\alpha_{i,t} : \Theta_{i}^t \times \Theta_{-i}^{t-1} \rightarrow X_{i,t}$

- Strategy also defines behavior following agent’s own deviations, but this is irrelevant for the normal form

- Strategy is *truthful-obedient* if for all $\theta^t$,

\[
\beta_{i,t}(\theta_{i}^t, \theta_{-i}^{t-1}) = \theta_{i,t},
\]
\[
\alpha_{i,t}(\theta^t) = \chi_{i,t}(\theta^t)
\]
Balanced Team Mechanism

- $U_i(\chi^*(\theta), \theta) = \sum_{t=1}^{\infty} \delta^t u_{i,t}(\chi_t(\tilde{\theta}^t), \tilde{\theta})$
Balanced Team Mechanism

- \( U_i(\chi^*(\theta), \theta) = \sum_{t=1}^{\infty} \delta^t u_{i,t} \left( \chi_t \left( \tilde{\theta}^t \right), \tilde{\theta} \right) \)

- Efficient decision \( \chi^* \): \( \max_{\chi} \mathbb{E}_{\tilde{\theta}}^{\mu[\chi]} \left[ \sum_i U_i(\chi^*(\theta), \theta) \right] \)
Balanced Team Mechanism

- \( U_i (\chi^*(\theta), \theta) = \sum_{t=1}^{\infty} \delta^t u_{i,t} \left( \chi_t \left( \tilde{\theta}^t \right), \tilde{\theta} \right) \)

- Efficient decision \( \chi^* : \max_{\chi} \mathbb{E}_{\theta}^{\mu[\chi]} \left[ \sum_i U_i (\chi^*(\theta), \theta) \right] \)

- Balanced Team Transfers:

\[
\psi_{i,t}^B (\theta^t) = \gamma_{i,t} (\theta_{i,t}, \theta^{t-1}) - \frac{1}{l-1} \sum_{j \neq i} \gamma_{j,t} (\theta_{j,t}, \theta^{t-1}), \text{ where}
\]

\[
\gamma_{j,t} (\theta_{j,t}, \theta^{t-1}) = \delta^{-t} \left( \mathbb{E}_{\tilde{\theta}}^{\mu_t[\chi]|\theta_{j,t}, \theta^{t-1}} \left[ \sum_{i \neq j} U_i (\chi^*(\theta), \theta) \right] \right) - \mathbb{E}_{\tilde{\theta}}^{\mu_t[\chi]|\theta^{t-1}} \left[ \sum_{i \neq j} U_i (\chi^*(\theta), \theta) \right]
\]
Balanced Team Mechanism

- \( U_i(\chi^*(\theta), \theta) = \sum_{t=1}^{\infty} \delta^t u_{i,t} \left( \chi_t \left( \tilde{\theta}^t \right), \tilde{\theta} \right) \)

- Efficient decision \( \chi^* : \max_{\chi} \mathbb{E}_{\tilde{\theta}}^\mu[\chi] \left[ \sum_i U_i(\chi^*(\theta), \theta) \right] \)

- Balanced Team Transfers:
  \[
  \psi^B_{i,t}(\theta^t) = \gamma_{i,t}(\theta_{i,t}, \theta^{t-1}) - \frac{1}{l-1} \sum_{j \neq i} \gamma_{j,t}(\theta_{j,t}, \theta^{t-1}), \text{ where }
  \gamma_{j,t}(\theta_{j,t}, \theta^{t-1}) = \delta^{-t} \left( \begin{array}{c}
  \mathbb{E}_{\tilde{\theta}}^\mu_{t}[\chi] | \theta_{j,t}, \theta^{t-1} [\sum_{i \neq j} U_i(\chi^*(\theta), \theta)] \\
  - \mathbb{E}_{\tilde{\theta}}^\mu_{t}[\chi] | \theta^{t-1} [\sum_{i \neq j} U_i(\chi^*(\theta), \theta)]
  \end{array} \right)
  \]

**Theorem**

Assume independent types: conditional on \( x^t_0 \), agent \( i \)'s private information \( \theta^t_i, x^t_i \) does not affect the distribution of \( \theta_{j,t} \), for \( j \neq i \). Also assume private values: \( u_{j,t}(x^t, \theta^t) \) does not depend on \( \theta^t_i, x^t_i \) for all \( t \), \( i \neq j \). Then balanced team mechanism is BIC.
Balancing: Example

In initial example:

\[-U_S (\chi(\hat{\theta}), \hat{\theta}_S) = c(\chi_1(\hat{\theta}_S), \hat{\theta}_S) + \delta c(\chi_2(\hat{\theta}_S, \hat{\theta}_{B,2}), \hat{\theta}_S) + \delta^2 c(\chi_3(\hat{\theta}_S, \hat{\theta}_{B,3}), \hat{\theta}_S)\]

\[\gamma_{B,3}(\hat{\theta}_{B,2}, \hat{\theta}_{B,3}, \hat{\theta}_S) = -c(\chi_3(\hat{\theta}_S, \hat{\theta}_{B,3}), \hat{\theta}_S) + \mathbb{E}_{\bar{\theta}_{B,3}} [c(\chi_3(\hat{\theta}_S, \bar{\theta}_{B,3}), \hat{\theta}_S) | \hat{\theta}_{B,2}]\]

\[\gamma_{B,2}(\hat{\theta}_{B,2}, \hat{\theta}_S) = -c(\chi_2(\hat{\theta}_S, \hat{\theta}_{B,2}), \hat{\theta}_S) - \delta \mathbb{E}_{\bar{\theta}_{B,3}} [c(\chi_3(\hat{\theta}_S, \bar{\theta}_{B,3}), \hat{\theta}_S) | \hat{\theta}_{B,2}]\]

\[+ \mathbb{E}_{\bar{\theta}_{B,2}, \bar{\theta}_{B,3}} [c(\chi_2(\hat{\theta}_S, \bar{\theta}_{B,2}), \hat{\theta}_S) + \delta c(\chi_3(\hat{\theta}_S, \bar{\theta}_{B,3}), \hat{\theta}_S)]\]
Balancing: Proof Sketch

Let $\Psi_j(\tilde{\theta}) = \sum_{i \neq j} U_i(\chi^*(\theta), \theta)$, pv of $j$’s payments:

$$
\delta^t \gamma_{j,t}(\hat{\theta}_j^t, \hat{\theta}_-^{t-1}) = \underbrace{\mathbb{E}_{\tilde{\theta}}^\mu_t[\chi] | \hat{\theta}_j,t, \hat{\theta}_1^{t-1}}_{\gamma_{j,t}^+(\hat{\theta}_j,t, \hat{\theta}_1^{t-1})} \underbrace{[\Psi_j(\tilde{\theta})]}_{\gamma_{j,t}^-(\hat{\theta}_1^{t-1})} - \underbrace{\mathbb{E}_{\tilde{\theta}}^\mu_t[\chi] | \hat{\theta}_1^{t-1}}_{\gamma_{j,t}^-(\hat{\theta}_1^{t-1})} \underbrace{[\Psi_j(\tilde{\theta})]}_{\gamma_{j,t}^+(\hat{\theta}_j,t, \hat{\theta}_-^{t-1})} $$

Two terms are expectations of the same function $\Psi_j(\tilde{\theta})$.

Claim 1: Expected present value of $\gamma_i, t$ equals, up to a constant, that of $\psi_i, t$

Claim 2: Expected present value of $\gamma_j, t$ is zero for each $j \neq i$.
Balancing: Proof Sketch

- Let $\Psi_j(\tilde{\theta}) = \sum_{i \neq j} U_i(\chi^*(\theta), \theta)$, pv of $j$’s payments:

$$\delta^t \gamma_{j,t}(\hat{\theta}_j^t, \hat{\theta}_{-j}^{t-1}) = \mathbb{E}_{\tilde{\theta}}^\mu_t[\chi]\hat{\theta}_{j,t},\hat{\theta}_t^{t-1} \left[ \Psi_j(\tilde{\theta}) \right] - \mathbb{E}_{\tilde{\theta}}^\mu_t[\chi]|\hat{\theta}_t^{t-1} \left[ \Psi_j(\tilde{\theta}) \right]$$

- Two terms are expectations of the same function $\Psi_j(\tilde{\theta})$
Balancing: Proof Sketch

- Let $\Psi_j(\tilde{\theta}) = \sum_{i \neq j} U_i(\chi^*(\theta), \theta)$, pv of j’s payments:

$$\delta^t \gamma_{j,t}(\hat{\theta}^t_j, \hat{\theta}^{t-1}_{-j}) = E^{|\theta^t_t[x]|}_{\hat{\theta}^t_j, \hat{\theta}^{t-1}_{-j}} [\Psi_j(\tilde{\theta})] - E^{|\theta^t_t[x]|}_{\hat{\theta}^{t-1}} [\Psi_j(\tilde{\theta})]$$

Two terms are expectations of the same function $\Psi_j(\tilde{\theta})$

- $\gamma^-_{j,t}(\hat{\theta}^{t-1})$ uses only period $t - 1$ information
Balancing: Proof Sketch

- Let $\Psi_j (\tilde{\theta}) = \sum_{i \neq j} U_i (\chi^*(\theta), \theta)$, pv of $j$'s payments:

$$\delta^t \gamma_{j,t} (\hat{\theta}_j, \hat{\theta}_{t-j}) = \mathbb{E}_{\hat{\theta}}^{\mu_t}[\chi] \gamma_{j,t}^+ (\hat{\theta}_j, \hat{\theta}_t) - \mathbb{E}_{\hat{\theta}}^{\mu_t}[\chi] \gamma_{j,t}^- (\hat{\theta}_t)$$

- Two terms are expectations of the same function $\Psi_j (\tilde{\theta})$
- $\gamma_{j,t}^- (\hat{\theta}_t)$ uses only period $t - 1$ information
- $\gamma_{j,t}^+(\hat{\theta}_j, \hat{\theta}_t)$ uses, in addition, agent $j$'s period-$t$ report
Balancing: Proof Sketch

- Let $\Psi_j(\tilde{\theta}) = \sum_{i \neq j} U_i(\chi^*(\theta), \theta)$, pv of $j$’s payments:

$$\delta^t \gamma_{j,t}(\tilde{\theta}_j, \tilde{\theta}^{t-1}_j) = \left[ E_{\tilde{\theta}}^{\mu_t} [\chi] | \hat{\theta}_{j,t}, \hat{\theta}^{t-1}_j \right] [\Psi_j(\tilde{\theta})] - \left[ E_{\tilde{\theta}}^{\mu_t} [\chi] | \hat{\theta}^{t-1}_j \right] [\Psi_j(\tilde{\theta})]$$

- Two terms are expectations of the same function $\Psi_j(\tilde{\theta})$

- $\gamma^-_{j,t}(\hat{\theta}^{t-1}_j)$ uses only period $t - 1$ information

- $\gamma^+_{j,t}(\hat{\theta}_{j,t}, \hat{\theta}^{t-1}_j)$ uses, in addition, agent $j$’s period-$t$ report

- For any deviation by agent $i$, if the others are truthful-obedient:
Let $\Psi_j(\tilde{\theta}) = \sum_{i \neq j} U_i(\chi^*(\theta), \theta)$, pv of j’s payments:

$$\delta^t \gamma_{j,t}(\hat{\theta}_j^t, \hat{\theta}_j^{t-1}) = \mathbb{E}_{\tilde{\theta}}^{\mu_t[\chi]|\hat{\theta}_j, t, \hat{\theta}_t^{t-1}} \left[ \Psi_j(\tilde{\theta}) \right] - \mathbb{E}_{\tilde{\theta}}^{\mu_t[\chi]|\hat{\theta}_t^{t-1}} \left[ \Psi_j(\tilde{\theta}) \right]$$

Two terms are expectations of the same function $\Psi_j(\tilde{\theta})$

$\gamma_{j,t}(\hat{\theta}_j^{t-1})$ uses only period $t-1$ information

$\gamma_{j,t}^+(\hat{\theta}_j, \hat{\theta}_j^{t-1})$ uses, in addition, agent j’s period-$t$ report

For any deviation by agent $i$, if the others are truthful-obedient:

- **Claim 1**: Expected present value of $\gamma_{i,t}$ equals, up to a constant, that of $\psi_{i,t}$
Let $\Psi_j(\tilde{\theta}) = \sum_{i \neq j} U_i(\chi^*(\theta), \theta)$, pv of $j$’s payments:

$$\delta^t \gamma_{j,t}(\hat{\theta}_j^t, \hat{\theta}_{-j}^{t-1}) = \mathbb{E}_{\tilde{\theta}}^{\mu_t[\chi]|\hat{\theta}_{j,t},\hat{\theta}_{-j}^{t-1}} [\Psi_j(\tilde{\theta})] - \mathbb{E}_{\tilde{\theta}}^{\mu_t[\chi]|\hat{\theta}_{-j}^{t-1}} [\Psi_j(\tilde{\theta})]$$

Two terms are expectations of the same function $\Psi_j(\tilde{\theta})$

- $\gamma_{j,t}^- (\hat{\theta}^{t-1})$ uses only period $t - 1$ information
- $\gamma_{j,t}^+ (\hat{\theta}_j^t, \hat{\theta}_{-j}^{t-1})$ uses, in addition, agent $j$’s period-$t$ report

For any deviation by agent $i$, if the others are truthful-obedient:

- **Claim 1:** Expected present value of $\gamma_{i,t}$ equals, up to a constant, that of $\psi_{i,t}$
- **Claim 2:** Expected present value of $\gamma_{j,t}$ is zero for each $j \neq i$
Proof of Claim 2

- For any possible deviation of agent $i$, expected present value of $\gamma_{j,t}$ is zero for each $j \neq i$:

\[
\begin{align*}
\theta_{j,1} & \quad \theta_{-j,1} & \quad \theta_{j,2} \\
-\gamma_{j,1}^- & \quad \gamma_{j,1}^+ & \quad -\gamma_{j,2}^- & \quad \gamma_{j,2}^+ & \quad \ldots & \quad \ldots \\
-\delta \gamma_{j,1} & \quad -\delta^2 \gamma_{j,2} & \quad -\delta^t \gamma_{j,t} & \quad \gamma_{j,t}^+
\end{align*}
\]
Proof of Claim 2

- For any possible deviation of agent $i$, expected present value of $\gamma_{j,t}$ is zero for each $j \neq i$:

- Independent types $\Rightarrow$ agent $i$’s private history $(\theta_i^t, x_i^{t-1})$ does not affect beliefs over $\tilde{\theta}_{j,t}$
Proof of Claim 2

- For any possible deviation of agent $i$, expected present value of $\gamma_{j,t}$ is zero for each $j \neq i$:

- Independent types $\Rightarrow$ agent $i$’s private history $(\theta_i^t, \chi_i^{t-1})$ does not affect beliefs over $\tilde{\theta}_{j,t}$

- If agent $j$ is truthful, the expectation of $\gamma_{j,t}^+(\tilde{\theta}_{j,t}, \hat{\theta}_t^{t-1})$ before time $t$ equals $\gamma_{j,t}^-(\hat{\theta}_t^{t-1})$, for any report history $\hat{\theta}_t^{t-1}$
Proof of Claim 2

- For any possible deviation of agent $i$, expected present value of $\gamma_{j,t}$ is zero for each $j \neq i$:

\[ \theta_{j,1} \quad \theta_{-j,1} \quad \theta_{j,2} \quad \ldots \quad \theta_{j,t} \]

\[ \begin{array}{c}
-\gamma_{j,1}^- \\
\delta \gamma_{j,1}^-
\end{array} \quad \begin{array}{c}
\gamma_{j,1}^+ \\
\delta^2 \gamma_{j,2}^-
\end{array} \quad \begin{array}{c}
-\gamma_{j,2}^- \\
\delta^t \gamma_{j,t}^-
\end{array} \quad \begin{array}{c}
\gamma_{j,2}^+ \\
\delta^t \gamma_{j,t}^+
\end{array} \quad \ldots \]

- Independent types $\Rightarrow$ agent $i$’s private history $(\theta_i^t, x_i^{t-1})$ does not affect beliefs over $\tilde{\theta}_{j,t}$

- If agent $j$ is truthful, the expectation of $\gamma_{j,t}^+(\tilde{\theta}_{j,t}, \hat{\theta}_{t}^{t-1})$ before time $t$ equals $\gamma_{j,t}^-(\hat{\theta}_{t}^{t-1})$, for any report history $\hat{\theta}_{t}^{t-1}$

- LIE: ex ante expectation of $\gamma_{j,t}$ equals zero
Proof of Claim 1

- For any possible deviation of agent $i$, expected present value of $\gamma_{i,t}$ equals, up to a constant, that of $\psi_{i,t}$:

$$\theta_{i,1} \quad \theta_{-i,1} \quad \ldots \quad \theta_{-i,t-1} \quad \theta_{i,t}$$

$$-\gamma_{i,1} \quad \gamma_{i,1} \quad -\gamma_{i,2} \quad \ldots \quad \gamma_{i,t-1} \quad -\gamma_{i,t} \quad \gamma_{i,t}$$

$$\leftarrow = 0 \rightarrow$$

Thus, expectation of $t \sum_{\tau=1}^{\delta} \tilde{\gamma}_{i,\tau}$ equals to that of $\tilde{\gamma}_{i} + \gamma_{i,t}$.
For any possible deviation of agent $i$, expected present value of $\gamma_{i,t}$ equals, up to a constant, that of $\psi_{i,t}$:

\[
\begin{align*}
\theta_{i,1} & \quad \theta_{-i,1} \\
-\gamma_{i,1} & \quad \gamma_{i,1} & -\gamma_{i,2} \\
\quad & \quad \text{=} 0 & \quad \text{=} 0
\end{align*}
\]

Independent types $\Rightarrow$ agent $i$’s private history $(\theta_{i,t}^t, x_{i,t-1}^t)$ does not affect beliefs over $\tilde{\theta}_{-i,t}$
Proof of Claim 1

- For any possible deviation of agent $i$, expected present value of $\gamma_{i,t}$ equals, up to a constant, that of $\psi_{i,t}$:

  \[
  \theta_{i,1} \quad \theta_{-i,1} \quad \ldots \quad \theta_{-i,t-1} \quad \theta_{i,t} \\
  \begin{array}{c}
  -\gamma_{i,1} \\
  \gamma_{i,1} \\
  -\gamma_{i,2} \\
  \gamma_{i,t-1} \\
  -\gamma_{i,t} \\
  \gamma_{i,t}
  \end{array}
  \]

  \[
  \begin{array}{c}
  \leftarrow = 0 \rightarrow \\
  \leftarrow = 0 \rightarrow
  \end{array}
  \]

- Independent types $\Rightarrow$ agent $i$’s private history $(\theta_{i,t}, x_{t-1}^{t-1})$ does not affect beliefs over $\tilde{\theta}_{-i,t}$

- If the others are truthful, agent $i$’s time-$t$ expectation of $\gamma_{i,t+1}(\tilde{\theta}_{-i,t}, \hat{\theta}_{i,t}, \hat{\theta}^{t-1})$ equals $\gamma_{i,t}(\tilde{\theta}_{i,t}, \hat{\theta}^{t-1})$ for any $\hat{\theta}_{i,t}, \hat{\theta}^{t-1}$
Proof of Claim 1

- For any possible deviation of agent \( i \), expected present value of \( \gamma_{i,t} \) equals, up to a constant, that of \( \psi_{i,t} \):

\[
\theta_{i,1} \quad \theta_{-i,1} \quad \ldots \quad \theta_{-i,t-1} \quad \theta_{i,t} \]

\[
\leftarrow = 0 \rightarrow
\]

\[
-\gamma_{i,1} \quad \gamma_{i,1} \quad \ldots \quad \gamma_{i,t-1} \quad \gamma_{i,t} \]

- Independent types \( \Rightarrow \) agent \( i \)'s private history \( (\theta^t_i, x^{t-1}_i) \) does not affect beliefs over \( \tilde{\theta}_{-i,t} \)

- If the others are truthful, agent \( i \)'s time-\( t \) expectation of \( \gamma_{i,t+1}(\tilde{\theta}_{-i,t}, \hat{\theta}_i, \hat{\theta}^{t-1}) \) equals \( \gamma_{i,t}(\hat{\theta}_i, \hat{\theta}^{t-1}) \) for any \( \hat{\theta}_i, \hat{\theta}^{t-1} \)

- LIE: the two terms have the same ex ante expectations as well
Proof of Claim 1

- For any possible deviation of agent $i$, expected present value of $\gamma_{i,t}$ equals, up to a constant, that of $\psi_{i,t}$:

$$\theta_{i,1} \quad \theta_{-i,1} \quad \ldots \quad \theta_{-i,t-1} \quad \theta_{i,t}$$

$$\gamma_{i,1}^+ \quad \gamma_{i,2}^- \quad \ldots \quad \gamma_{i,t-1}^+ \quad \gamma_{i,t}^-$$

- Independent types $\Rightarrow$ agent $i$’s private history $(\theta_i^t, x_i^{t-1})$ does not affect beliefs over $\tilde{\theta}_{-i,t}$
- If the others are truthful, agent $i$’s time-$t$ expectation of $\gamma_{i,t+1}^- (\tilde{\theta}_{-i,t}, \hat{\theta}_{i,t}, \hat{\theta}^{t-1})$ equals $\gamma_{i,t}^+ (\hat{\theta}_{i,t}, \hat{\theta}^{t-1})$ for any $\hat{\theta}_{i,t}, \hat{\theta}^{t-1}$
- LIE: the two terms have the same ex ante expectations as well
- Thus, expectation of $\sum_{\tau=1}^{t} \delta^\tau \tilde{\gamma}_{i,\tau}$ equals to that of $\tilde{\gamma}_{i,t}^+ - \tilde{\gamma}_{i,1}^-$
Proof of Claim 1

For any possible deviation of agent $i$, expected present value of $\gamma_{i,t}$ equals, up to a constant, that of $\psi_{i,t}$:

\[
\theta_{i,1} \quad \theta_{-i,1} \quad \theta_{-i,t-1} \quad \theta_{i,t} \\
\gamma_{i,1}^{+} \quad \gamma_{i,2}^{-} \quad \gamma_{i,t-1}^{+} \quad \gamma_{i,t}^{+}
\]

\[\gamma^{+}_{i,t} \left( \tilde{\theta}_{-i,t}, \hat{\theta}_{i,t}, \hat{\theta}^{t-1} \right) = \gamma^{-}_{i,t} \left( \tilde{\theta}_{i,t}, \hat{\theta}_{i,t}, \hat{\theta}^{t-1} \right) \]

\[\gamma_{i,1}^{-} \quad \gamma^{+}_{i,1} \quad \gamma^{+}_{i,t} \quad \gamma^{-}_{i,t} \quad \gamma_{i,t}^{-}
\]

- Independent types $\Rightarrow$ agent $i$’s private history $(\theta_{i}^{t}, x_{i}^{t-1})$ does not affect beliefs over $\tilde{\theta}_{-i,t}$
- If the others are truthful, agent $i$’s time-$t$ expectation of $\gamma_{i,t+1}^{-}(\tilde{\theta}_{-i,t}, \hat{\theta}_{i,t}, \hat{\theta}^{t-1})$ equals $\gamma_{i,t}^{+}(\tilde{\theta}_{i,t}, \hat{\theta}_{i,t}, \hat{\theta}^{t-1})$ for any $\hat{\theta}_{i,t}, \hat{\theta}^{t-1}$
- LIE: the two terms have the same ex ante expectations as well

Thus, expectation of $\sum_{\tau=1}^{t} \delta^{\tau} \tilde{\gamma}_{i,\tau}$ equals to that of $\tilde{\gamma}_{i,t}^{+} - \tilde{\gamma}_{i,1}^{-}$

- $\gamma_{i,1}^{-}$ is unaffected by reports; $\tilde{\gamma}_{i,t}^{+} \rightarrow \Psi_{i} (\tilde{\theta})$ as $t \rightarrow \infty$
Decentralized Games (No External Enforcer)

- In each period $t = 1, 2, ...$
Decentralized Games (No External Enforcer)

- In each period $t = 1, 2, ...$
  - Each agent $i$ privately observes signal $\theta_{i,t}$
Decentralized Games (No External Enforcer)

- In each period $t = 1, 2, ...$
  1. Each agent $i$ privately observes signal $\theta_{i,t}$
  2. Agents send simultaneous reports

Markovian Assumptions:
- Finite action, type spaces, the same in each period
- Markovian type transitions: $\nu_t(\theta_t, x_t) = \bar{\nu}(\theta_t, x_t)$
- Stationary separable payoffs $u_{i,t}(x_t, \theta_t) = \bar{u}_i(x_t)$

"Blackwell policy" $\chi$ - a Markovian decision rule that is efficient for all $\delta$ close enough to 1, for any starting state
Decentralized Games (No External Enforcer)

- In each period $t = 1, 2, ...$
  1. Each agent $i$ privately observes signal $\theta_{i,t}$
  2. Agents send simultaneous reports
  3. Each agent $i$ chooses private action $x_{i,t}$
Decentralized Games (No External Enforcer)

- In each period \( t = 1, 2, \ldots \)
  1. Each agent \( i \) privately observes signal \( \theta_{i,t} \)
  2. Agents send simultaneous reports
  3. Each agent \( i \) chooses private action \( x_{i,t} \)
  4. Each agent \( i \) chooses public action \( x_{0,i,t} \), makes public payment \( z_{i,j,t} \geq 0 \) to each agent \( j \)
Decentralized Games (No External Enforcer)

- In each period $t = 1, 2, ...$
  1. Each agent $i$ privately observes signal $\theta_{i,t}$
  2. Agents send simultaneous reports
  3. Each agent $i$ chooses private action $x_{i,t}$
  4. Each agent $i$ chooses public action $x_{0,i,t}$, makes public payment $z_{i,j,t} \geq 0$ to each agent $j$

  $\Rightarrow$ Public action $x_{0,t} = (x_{0,i,t})_{i=1}^N$, total transfer $y_{i,t} = \sum_j (z_{j,i,t} - z_{i,j,t})$ to agent $i$ (budget-balanced)
Decentralized Games (No External Enforcer)

- In each period \( t = 1, 2, \ldots \)
  1. Each agent \( i \) privately observes signal \( \theta_{i,t} \)
  2. Agents send simultaneous reports
  3. Each agent \( i \) chooses private action \( x_{i,t} \)
  4. Each agent \( i \) chooses public action \( x_{0,i,t} \), makes public payment \( z_{i,j,t} \geq 0 \) to each agent \( j \)

\[ \Rightarrow \text{Public action } x_{0,t} = (x_{0,i,t})_{i=1}^{N}, \text{ total transfer } y_{i,t} = \sum_{j} (z_{j,i,t} - z_{i,j,t}) \text{ to agent } i \text{ (budget-balanced)} \]

- Markovian Assumptions:
  - Finite action, type spaces, the same in each period
  - Markovian type transitions:
    \[ \nu_{t} = \bar{\nu}(\theta_{t,j}, \theta_{t,1}, x_{t,1}) \]
  - Stationary separable payoffs
    \[ u_{i,t}(x_{t}, \theta_{t}) = \bar{u}_{i}(x_{t}, \theta_{t}) \]
  - "Blackwell policy" \( \chi \)- a Markovian decision rule that is efficient for all \( \delta \) close enough to 1, for any starting state

Can we sustain \( \chi \) in PBE?
Decentralized Games (No External Enforcer)

- In each period \( t = 1, 2, ... \)
  1. Each agent \( i \) privately observes signal \( \theta_{i,t} \)
  2. Agents send simultaneous reports
  3. Each agent \( i \) chooses private action \( x_{i,t} \)
  4. Each agent \( i \) chooses public action \( x_{0,i,t} \), makes public payment \( z_{i,j,t} \geq 0 \) to each agent \( j \)

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\[ \Rightarrow \text{Public action} \ x_{0,t} = (x_{0,i,t})_{i=1}^{N}, \text{total transfer} \]
\[ y_{i,t} = \sum_{j} (z_{j,i,t} - z_{i,j,t}) \] to agent \( i \) (budget-balanced)

- Markovian Assumptions:
  - Finite action, type spaces, the same in each period
  - Markovian type transitions: \( \nu_t \left( \theta_t | \theta_{t-1}, x_{t-1} \right) = \bar{v} \left( \theta_t | \theta_{t-1}, x_{t-1} \right) \)
Decentralized Games (No External Enforcer)

- In each period $t = 1, 2, ...$
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  2. Agents send simultaneous reports
  3. Each agent $i$ chooses private action $x_{i,t}$
  4. Each agent $i$ chooses public action $x_{0,i,t}$, makes public payment $z_{i,j,t} \geq 0$ to each agent $j$

$\Rightarrow$ Public action $x_{0,t} = (x_{0,i,t})_{i=1}^N$, total transfer $y_{i,t} = \sum_j (z_{j,i,t} - z_{i,j,t})$ to agent $i$ (budget-balanced)

- Markovian Assumptions:
  - Finite action, type spaces, the same in each period
  - Markovian type transitions: $\nu_t (\theta_t | \theta_t^{t-1}, x_t^{t-1}) = \bar{\nu} (\theta_t | \theta_{t-1}, x_{t-1})$
  - Stationary separable payoffs $u_{i,t} (x_t, \theta_t) = \bar{u}_i (x_t, \theta_t)$
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- $\Rightarrow \exists$ a “Blackwell policy” $\chi^*$ - a Markovian decision rule that is efficient for all $\delta$ close enough to 1, for any starting state
Decentralized Games (No External Enforcer)

- In each period $t = 1, 2, ...$
  1. Each agent $i$ privately observes signal $\theta_{i,t}$
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\[ \Rightarrow \text{Public action } x_{0,t} = (x_{0,i,t})_{i=1}^N, \text{ total transfer} \]
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  - Finite action, type spaces, the same in each period
  - Markovian type transitions: $\nu_t \left( \theta_t | \theta^{t-1}, x^{t-1} \right) = \bar{\nu} \left( \theta_t | \theta_{t-1}, x_{t-1} \right)$
  - Stationary separable payoffs $u_{i,t} (x^t, \theta^t) = \bar{u}_i (x_t, \theta_t)$

\[ \Rightarrow \exists \text{ “Blackwell policy” } \chi^* \text{ - a Markovian decision rule that is efficient for all } \delta \text{ close enough to 1, for any starting state} \]

- Can we sustain $\chi^*$ in PBE?
Implement the Balanced Team Mechanism

When no publicly observed deviation, make payments

\[ z_{i,j,t} = \frac{1}{l-1} \gamma_{j,t}(\theta_j^t, \tilde{\theta}_{-j}^{t-1}) + K_i \]

\[ = \frac{1}{l-1} \sum_{k \neq j} \sum_{\tau=t}^{\infty} \delta^{\tau-t} \left( \mathbb{E}_{\tilde{\theta}}^{\mu_t[X^*]|\theta_j^t,\tilde{\theta}_{-j}^{t-1}} \right) \left[ \tilde{u}_k (X^* (\tilde{\theta}_\tau), \tilde{\theta}_\tau) \right] + K_i \]
Implement the Balanced Team Mechanism

When no publicly observed deviation, make payments

\[
    z_{i,j,t} = \frac{1}{l-1} \gamma_{j,t}(\theta^t_j, \theta^{t-1}_j) + K_i
\]

\[= \frac{1}{l-1} \sum_{k \neq j} \sum_{\tau=t}^\infty \delta^{\tau-t} \left( \mathbb{E}_{\tilde{\theta}}^{\mu_t}[\chi^*]|\theta^t_j,\theta^{t-1}_j \right. \left[ \bar{u}_k (\chi^* (\tilde{\theta}_{\tau}), \tilde{\theta}_{\tau}) \right) + K_i \]

Can we prevent public deviations (="quitting") for any history?
Implement the Balanced Team Mechanism

- When no publicly observed deviation, make payments

\[ z_{i,j,t} = \frac{1}{l-1} \gamma_{j,t}(\theta^t_j, \theta^{t-1}_j) + K_i \]

\[ = \frac{1}{l-1} \sum_{k \neq j} \sum_{\tau=t}^{\infty} \delta^{\tau-t} \left( \mathbb{E}_{\hat{\theta}}^{\mu_t} [\chi^*] | \theta^t_j, \theta^{t-1}_j \left[ \tilde{u}_k (\chi^* (\tilde{\theta}_\tau), \tilde{\theta}_\tau) \right] \right) + K_i \]

- Can we prevent public deviations (= “quitting”) for any history?
  - Can think of this as joint IC-IR constraints
Implement the Balanced Team Mechanism

- When no publicly observed deviation, make payments

\[ z_{i,j,t} = \frac{1}{l-1} \gamma_{j,t}(\theta_j^t, \theta_{-j}^{t-1}) + K_i \]

- Can we prevent public deviations (=“quitting”) for any history?
  - Can think of this as joint IC-IR constraints
- Problem: transfers may be unbounded as \( \delta \to 1 \).
Implement the Balanced Team Mechanism

When no publicly observed deviation, make payments

\[ z_{i,j,t} = \frac{1}{l-1} I \gamma_{j,t}(\theta_{j}^{t}, \theta_{-j}^{t-1}) + K_{i} \]

\[ = \frac{1}{l-1} \sum_{k \neq j} \sum_{\tau = t}^{\infty} \delta^{\tau-t} \left( \begin{array}{c} \mathbb{E}^{\mu_{t}[\chi^{*}]}_{\bar{\theta}}[\tilde{\theta}_{\tau}, \tilde{\theta}_{\tau}] \left[ \tilde{u}_{k} (\chi^{*} (\tilde{\theta}_{\tau}), \tilde{\theta}_{\tau}) \right] \\ -\mathbb{E}^{\mu_{t}[\chi^{*}]}_{\bar{\theta}}[\theta_{t-1}^{\tau}] \left[ \tilde{u}_{k} (\chi^{*} (\tilde{\theta}_{\tau}), \tilde{\theta}_{\tau}) \right] \end{array} \right) + K_{i} \]

Can we prevent public deviations (= “quitting”) for any history?

- Can think of this as joint IC-IR constraints
- Problem: transfers may be unbounded as \( \delta \to 1 \).
- But: with limited persistence of \( \bar{\theta} \), the two expectations may be close as \( \tau \to \infty \)
Theorem

Take the Markov game with independent private values, which has a zero-payoff belief-free static NE. Suppose that a Blackwell policy \( \chi^* \) induces a Markov process with a unique ergodic set (and a possibly empty transient set), and that the ergodic distribution gives a positive expected total surplus. Then for \( \delta \) large enough, \( \chi^* \) can be sustained in a PBE using Balanced Team Transfers.
- Dynamic Games

  - In decentralized games, actions and transfers have to be self-enforcing; not commitment mechanism is available to the agents
  - In many games, transfers are not available
  - What is the relationship between the outcomes that can be attained WITH commitment and transfers, and what can be attained without?
  - When can efficiency be sustained as an eqm?
  - What do equilibria look like for different discount factors?
  - Efficiency includes BB
• Literature in Microeconomics on Dynamic Games and Contracts
  – Collusion: Athey and Bagwell (series of papers)
  – Repeated Trade: Athey and Miller
  – Relational Contracts: Levin, Rayo
  – Continuous time models, principal agent: Sannikov and coauthors
  – Cost of ex post as opposed to Bayesian equilibrium: Miller

• Literature in Dynamic Public Finance, Macro
  – Amador, Angeletos, and Werning; Tsyvinski; Athey, Atkeson, and Kehoe; others
Focus Today: Hidden Information

- Hidden actions impt, techniques and applications often different
- Auctions, collusion, bilateral or multilateral trade, public good provision, resource allocation, favor-trading in relationships, mutual insurance

Contracts, Games, and Games as Contracts
Mechanism Design Approach to Dynamic Games

- In static theory, we are familiar with mechanism design approach to analyzing games such as auctions
- Use tools such as envelope theorem, revenue equivalence, etc. to characterize equilibria
- Analyze constraints
- Take this approach to dynamic games
- Combine dynamic programming and mechanism design tools
- Frontier of current research: fully dynamic games (not repeated)
A Toolkit for Analyzing Dynamic Games and Contracts

- Abreu-Pearce-Stacchetti and dynamic programming
- The mechanism design approach to repeated games with hidden information
- Sustaining efficiency with transfers
- The folk theorem without transfers
- Dynamic Programming for Dynamic Games
Analyzing Repeated and Dynamic Games with Hidden Information

- Model the game/contract in extensive form
  - Dynamic games—see Battiglini (2005), Athey and Segal (2007)
  - Cumbersome to specify full strategy space and optimize over it

- Use APS/Mechanism Design combination
  - Applicability of results with the right assumptions
  - Can apply body of knowledge for hidden info games
A Dynamic Game with Time-Varying Hidden Information

- Players $i = 1, \ldots, I$
- Time $t = 1, \ldots, T$ (special cases: $T = 1, T = \infty$)
- Superscript/subscript notation: given $((y_{i,t})_{t=1}^{T})_{i=1}^{I}$,
  $y_t = (y_{i,t})_{i=1}^{I}$, $y_i = (y_{i,t})_{t=1}^{T}$, $y^t = (y^t_{t'})_{t'=1}^{t}$.
- Type spaces $\Theta_{i,t} \subseteq \mathbb{R}^n$, random variables $\bar{\theta}_{i,t}$ with realizations $\theta_{i,t}$.
- Communication among players: $m_{i,t} \in \mathcal{M}_{i,t}$
- Decisions $X_{i,t} \subseteq \mathbb{R}^n$.
- Transfer from player $j$ to player $i$:
  $y_{j,i,t} \geq 0$, let $y_{i,t} = \sum_j y_{j,i,t} - y_{i,j,t}$.
  - Some models rule out transfers, e.g. collusion
History has two components:

- Public history $h^{t-1} = (x^{t-1}, m^{t-1}, y^{t-1})$, private histories $\theta^{t-1}$

Timeline in period $t$:

- Types realized ($\theta_t$)
  * History potentially affects distributions: $F_t(\theta_t; x^{t-1}, \theta^{t-1})$.
- Players communicate ($m_t$)
- Players simultaneously make decisions ($x_t$) and send transfers ($y_t$)

Note: can consider models without communication in this framework

- Messages can be contentless
- Athey-Bagwell (2001) show this can relax incentive constraints
Approach: Model Game with Mechanism Design Tools

- Define a recursive (direct revelation) mechanism
  - Replace mapping from types to actions with reporting strategy
  - Many games of interest have single crossing property, already restricted to monotone strategies

- Specify appropriate constraints
  - “On-schedule” and “off-schedule” deviations
  - Comparison between decentralized game and recursive mechanism
    * Game has add’l constraints, action space unrestricted
    * With patience, these can be satisfied
    * Game without transfers must deal with restrictions on continuation values
• The role of patience
  – Static mechanism that satisfies BIC, EPBB, IR may not be eqm in decentralized game with low patience
    * Mechanism provides commitment
  – Static mechanism that satisfies BIC, EPBB, fails IR may be eqm in game with high patience
    * Future gain from relationship relaxes participation constraints
• Independent (over time) types or perfectly persistent types
  – Use static tools
• More general dynamics
  – Contingent, multi-stage deviations
  – Transfers and continuation equilibria not perfect substitutes
Approach Here: Recursive Mechanisms

  - Miller (2005) sets out approach for general model

- Idea: use APS approach together with mechanism design tools

- Start by focusing on stationary (repeated) games
  - For appropriately selected constraints, a “self-generating” recursive mechanism will be a PPE
  - A PPE can be written as a recursive mechanism

- Apply tools from static mechanism design theory
The Recursive Mechanism

- Stage Mechanism
  - Action plan for each player: $\chi : \Theta_t \to X$
  - Transfer plan from $i$ to $j$, $\psi_{i,j} : \Theta_t \to \mathbb{R}^+$, $\psi_i = \sum_j \psi_{j,i} - \psi_{i,j}$
  - Continuation value function $w : \Theta_t \to \mathbb{R}^I$.
  - Let $\gamma = (\chi, \psi, w)$
  
  Ex post utility: $u_i(\hat{\theta}_t, \theta_{i,t}; \gamma) = \pi_i(\chi(\hat{\theta}_t), \theta_{i,t}) + \psi_i(\hat{\theta}_t) + \delta w_i(\hat{\theta}_t)$

  Interim utility: $\bar{u}_i(\hat{\theta}_{i,t}, \theta_{i,t}; \gamma) = \mathbb{E}_{\tilde{\theta}_{-i,t}}[u_i((\hat{\theta}_{i,t}, \tilde{\theta}_{-i,t}), \theta_{i,t}; \gamma)]$

- Recursive Mechanism: $\langle V, \{\gamma(v)\}_{v \in V}, v_0 \rangle$
  - A set $V$ An initial condition $v_0 \in V$
  - A set of stage mechanisms $\{\gamma(v)\}_{v \in V}$
Constraints

- (Bayesian, Interim) IC:
  \[ \bar{u}_i(\theta_{i,t}, \theta_{i,t}; \gamma) \geq \bar{u}_i(\hat{\theta}_{i,t}, \theta_{i,t}; \gamma) \text{ for all } \hat{\theta}_{i,t} \in \Theta_{i,t} \]

- IR\((p_0)\)
  - “Outside option”: punishment equilibrium with payoffs \(p_0\).
  - Could be static Nash, “Nonparticipation.”
  - For simplicity, assume informative communication.

\[
\bar{u}_i(\theta_{i,t}, \theta_{i,t}; \gamma) \geq \sup_{\hat{\theta}_{i,t}} \left\{ \mathbb{E}_{\hat{\theta}_{-i,t}} \left[ \sup_{x_i} \left( \pi_i(x_i, x_{-i}(\hat{\theta}_{i,t}, \bar{\theta}_{-i,t}), \theta_{i,t}) + \sum_j \psi_{j,i}(\hat{\theta}_{i,t}, \bar{\theta}_{-i,t}) \right) \right] \right\} + \delta p_{0,i}.
\]

* More generally, take expectations given messages. See Athey and Bagwell (2001) for more discussion of alternative IRs.
* Note assn about transfers and actions simultaneous.
Self-Generating Recursive Mechanism

- Define the set of attainable payoffs to be

$$\mathcal{V} = \text{co} \left\{ v \in \mathbb{R}^I : \exists \gamma \text{ s.t. } \sum_i v_i = \sum_i \frac{\mathbb{E}_{\tilde{\theta}_t} \left[ u_i(\tilde{\theta}_t, \tilde{\theta}_{i,t}; \gamma) \right]}{1 - \delta} \right\}.$$
• For $V \subseteq \mathcal{V}$, $p_0 \in \mathbb{R}^I$, define $T(V; p_0)$ to be the set of $v \in \mathbb{R}^I$ for which there exist stage mechanisms $\gamma(v) = (\chi, \psi, w)(v)$ whereby

1. Promise-keeping: $\mathbb{E}_{\tilde{\theta}_t} \left[ u_i(\tilde{\theta}_t, \tilde{\theta}_i, t; \gamma(v)) \right] = v_i$.
2. Coherence: $w(v) : \Theta_t \to V$.
3. Best response: $\gamma(v)$ satisfies IC and IR($p_0$).

• $V$ is self-generating relative to $p_0$ if $V \subseteq T(V; p_0)$.

  – Note: full set is $V \cup p_0$. Worst eqm not our focus; can extend to address this.

• $\langle V, \{\gamma(v)\}_{v \in V}, v_0 \rangle$ is self-generating relative to $p_0$ (SGRM($p_0$)) if:
  $V$ is self-generating relative to $p_0$ and,
  for each $v \in V$, (1)-(3) hold for $\gamma(v)$ and $p_0$. 

Proposition 1  Fix δ. Suppose \( p_0 \) is a PPE and consider \( V >> p_0 \).

(i) If \( V \) is a set of PPE payoffs with informative communication, then there exists \( v_0 \in V, \{\gamma(v)\}_{v \in V} \) such that \( \langle V, \{\gamma(v)\}_{v \in V}, v_0 \rangle \) is a SGRM\((p_0)\).

(ii) Suppose that \( \langle V, \{\gamma(v)\}_{v \in V}, v_0 \rangle \) is a SGRM\((p_0)\). Then \( V \) is in the set of PPE payoffs.

- Proof: See Miller (2005) (does folk theorem; adapt arguments). Analogous to APS. Have to verify that constraints deter relevant deviations.

- If interested in set \( V \) of PPE payoffs w/o informative communication, modify IRs to get corresponding result.

- IR constraints imply that deviating “off-schedule” is not desirable.
Transforming to a Static Problem: The Case with Transfers

• Recall

\[ u_i(\hat{\theta}_t, \theta_{i,t}; \gamma) = \pi_i(\chi(\hat{\theta}_t), \theta_{i,t}) + \psi_i(\hat{\theta}_t) + \delta w_i(\hat{\theta}_t). \]

  – With independent types, value for future play is the same for all types
  – Transfers and continuation values completely fungible

• WLOG, can consider stationary mechanisms (Levin, 2003)

• Then, consider static mechanism design problem with bounds on transfers imposed by IR
Proposition 2 Given $\chi$, suppose there exist EPBB, uniformly bounded, IC transfers for $\chi$, and that

$$\sum_i \mathbb{E}[\pi_i(\chi(\theta_t), \theta_{i,t})] > \sum p_{0,i}.$$ 

Then for $\delta$ sufficiently large, there exists a $SGRM(p), \langle V, \{\gamma(v)\}_{v \in V}, v_0 \rangle$ that is stationary, where

$$\sum_i v_{0,i} = \sum_i \mathbb{E}[\pi_i(\chi(\theta_t), \theta_{i,t})].$$

- Result says that if policy can be implemented with commitment, it can be self-enforcing for sufficiently patient agents.
• As $\delta$ grows, value of future eventually outweighs transfers. Independent future key.
Transforming to a Static Problem: The Case without Transfers

- Continuation values can mimic role of transfers, but for fixed $\delta$, Pareto frontier of $V$ is not in general linear
- Tradeoff between using variation in continuation values to provide incentives, and Pareto efficient continuation values
  - “Efficiency today v. efficiency tomorrow”
  - Finding: Sacrifice efficiency today
- Details of model determine shape of frontier of $V$
  - Multiplicity of efficient outcomes: partial linearity
- Approach (see Athey and Bagwell (2001)): start with large $V$, characterize $T(V)$
  - Analogous to static problem with restricted transfers
Folk Theorem without Transfers

  - Small changes in future per-period utility mimic transfers
  - FLM make unnecessary assumptions: independent, finite types
    * They focus on hidden action models and so don’t look for most general conditions
  - Miller (2005) generalizes to continuous types, correlated values

- Key elements of argument
  - Angle of supporting hyperplanes doesn’t matter generically
  - Average period payoffs (outside set) and hyperplane (inside set)
  - As $\delta \to 1$, length of hyperplane shrinks fast enough
  - Nothing about what to do for fixed $\delta$
FIGURE: Supporting Hyperplanes

\[(1-\delta)\pi + \delta w = v\]
Applications

- Ongoing Relationships
  - Time-varying individual costs and benefits to acting, i.i.d. private information
  - Restrictions on monetary transfers

- Examples
  - Colluding firms, i.i.d. cost/inventory shocks
  - Public good provision
    * Families/villages
    * Legislatures
  - Policy games (government is privately informed)
• Questions about Collusion
  – Response of collusive behavior to institutional setting
  – Effects of anti-trust policy (Restrictions on communication, side-payments)
  – Market design: info. about indiv. bids and identities
  – Institutional design: industry assoc., smoke-filled rooms

• Central Tradeoffs
  – Productive efficiency requires low-cost firm serves market
  – Firms like market-share, incentive to mimic low-cost firm
  – Need low prices or future “punishment” with high market-share
  – Future price wars v. “future market-share favors”
Asymmetric Collusion

• Setup
  – 2 firms produce perfect substitutes
  – Unit mass of consumers, reservation price $r$
  – 2 cost types: $\theta^i \in \{\theta_L, \theta_H\}$, $\Pr(\theta^i = \theta_j) = \eta_j$.
    Case: $\eta_L > 1/2$.

• Firms...
  – may split the market unevenly; details not imp’t.
  – may not charge different prices to different consumers.
  – communicate prior to producing (see Athey and Bagwell (2001) for analysis of communication)
Summary of Ideas for Asymmetric Eq’a

- A first best scheme, always price at $r$
  - Eqm described by two “states”
  - Each period, announce types
  - State $x$: low cost firm serves market, but firm 2 serves most of market if firms have same cost
    * If $(H, L)$, switch to state $y$, oth. return to $x$
  - State $y$: low cost firm serves market, but firm 1 serves most of market if firms have same cost
    * If $(L, H)$, switch to state $x$, oth. return to $y$

- Paper: shows that first-best scheme can work if patient enough that diff. betw. $x$ and $y$ provides suff. incentives; if less patient shows similar schemes with partial prod. eff. are optimal.
Illustration of First-Best equilibrium
A Linear Self-Generating Set with First-Best Profits

- Goal: Compute a critical discount factor above which first-best profits can be attained in every period.

  - Requires linear, “self-generating” set with slope $-1 :$

    $$ [(x, y), (y, x)] $$

  - Two parts.

  - “Adding Up”: First, ignore IC-Off. Is it possible to have linear self-generating set with full efficiency?

    * Need to implement $(x, y)$ using $v_{jk} \in [(x, y), (y, x)]$.

    * Future looks brighter than today for firm 1, and enough brighter when firm 1 has high cost to satisfy IC-On.
* Does it all “add up”? 

- Second, when are IC-Off’s cleared?

**Proposition 3** Suppose that $r - \theta_H < \theta_H - \theta_L$. Then, for all $\delta \in (\delta^FB, 1]$, there exist values $y > x > 0$ such that $x + y = 2\pi^{FB}/(1 - \delta)$, and the line segment $[(x, y), (y, x)]$ is “self-generating” and in the set of PPE values, $V^*$. 
Persistent Types

• See Cole and Kocherlakota, Athey and Bagwell on persistent types and extending recursive mechanism design approach

• Two-period sophisticated rotation
  – Produce today, give up market share tomorrow
  – Not very effective with persistent types

• First-best example
  – Extends to persistent types
  – Keep track of beliefs as state variables
  – In a fully revealing equilibrium, all that matters is last period’s state
As persistence grows relative to patience, rigid pricing approximately optimal with log-concavity

– Cannot do efficient transfers, so pooling is optimal
FIGURE: First-Best Equilibrium with Persistent Types
Summing Up Dynamic Games

- Bring together mechanism design and dynamic programming to analyze repeated and dynamic games
- Apply tools from static literature
- Generalize to incorporate interesting dynamics
  - Today: Serial correlation
  - Learning-by-doing, experimentation, information gathering (Athey-Segal)
  - Maintaining budget account (Athey-Miller)
- Efficiency possible in wide range of circumstances
- Pooling is optimal for agents when limited instruments for providing incentives