Borrowing in the Limit as our Nerdiness Goes to Infinity

DANIEL REEVES
Beeminder, LLC

Suppose I owe you n payments of S/n dollars—total owed is S if there were no time-discounting—spread evenly over an amount of time t, at interest rate r per unit time. For example, I might owe you \$120 in three payments over the coming year: \$40 three months from now, another \$40 in six months, and another \$40 in nine months. If I wanted to instead just pay you once, a full \$120, when should I do so to be perfectly equivalent to the three spread-out payments, given the interest rate?

Now let's generalize! Maybe I owe you \$120/year for a year of some continuous service and we figure it would be more fair to make monthly payments of \$10 instead of quarterly. And then fairer still would be smaller daily payments. And then, being the theoretically-minded nerds we are, why not work it out in the limit of exquisitely perfect fairness, where I pay you continuously? That being impractical though, what's the mathematical equivalent of that with a single payment? At what time should I pay you a lump sum of \$$S$ to be equivalent to a continuous stream of infinitesimal payments totaling \$$S$, spread over time t? In other words, what is the answer to the original question in the limit as n goes to infinity (and with continuous compounding)? And finally, what's interesting (groan) about the answer in terms of real-world application?

Send solutions to the puzzle editor at `dreeves@umich.edu` with subject: `nerdloan`. The author(s) of the most elegant solution (as judged by the editor) will be allowed to publish it in the next issue of the Exchanges (ties broken in favor of earlier submissions). To make the solutions accessible to a wide audience, please try to minimize technical jargon. Until the winner is chosen the editor will not give any hints or feedback.

Thanks to Yiling Chen and Felix Fischer for trying out this puzzle and suggesting improvements.

Author's address: `dreeves@umich.edu`