Algorithmic Information Structure Design: A Survey

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Information structure design, also sometimes known as *signaling* or *persuasion*, is concerned with understanding the effects of information on the outcomes of strategic interactions (the descriptive question), and in characterizing and computing the information sharing strategies which optimize some design objective (the prescriptive question). Both questions are illuminated through the lens of algorithms and complexity, as evidenced by recent work on the topic in the algorithmic game theory community. This monograph is a biased survey of this work, and paints a picture of the current state of progress and challenges ahead.

We divide information structure design into *single agent* and *multiple agent* models, and further subdivide the multiple agent case into the *public channel* and *private channel* modes of information revelation. In each of these three cases, we describe the most prominent models and applications, survey the associated algorithms and complexity results and their structural implications, and outline directions for future work.

Categories and Subject Descriptors: F.0 [**Theory of Computation**]: General

General Terms: Algorithmic, Economics, Theory

Additional Key Words and Phrases: Signaling, Persuasion, Information Structures

1. INTRODUCTION

There are two primary ways of influencing the behavior of self-interested agents: by providing *incentives*, or by influencing *beliefs*. The former is the domain of traditional mechanism design, and involves the promise of tangible rewards such as goods and/or money. The latter, the focus of this survey and the subject of a recent flurry of interest in both the computer science and economics communities, involves the selective provision of payoff-relevant information to agents through strategic communication. Such "sweet talk" was estimated by [McCloskey and Klamer 1995] to account for a quarter of all economic activity in the United States in 1995, and the estimate has since been revised to 30% [Antioch 2013]. This is emblematic of the emergence of large-scale social and economic networks, with countless transactions and interactions among asymmetrically-informed parties occurring daily.

The primary object of interest in this topic is the *information structure* of a game of incomplete information. Informally, the information structure determines "who knows what" about the payoff structure of the game, and in doing so determines the set of equilibria. More formally, an information structure maps the *state of nature* θ — a parameter or set of parameters which determines the payoff function of the game — to a *signal* for each agent in the game. The map is typically randomized, and therefore reveals partial and noisy information regarding the payoffs

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of various strategies. Like traditional incentive mechanisms, information structures can be studied either descriptively or prescriptively. The latter concerns the task faced by a *principal* who can control agents' access to information and wishes to optimize some objective function at the resulting equilibrium. Also like in mechanism design, the prescriptive question is naturally algorithmic, and studying it with the computational lens both provides structural insights and paves the way towards application. The task of optimizing the information structure in order to further an objective is often referred to as *information structure design*, *signaling*, or *persuasion*. In this survey, we refer to an information structure as a *signaling scheme* when we wish to think of it as a randomized algorithm — implemented by the principal — which takes as input a state of nature and outputs a signal for each agent.

1.1 The Basic Model and Assumptions

We use Θ to denote the family of states of nature, and assume $\theta \in \Theta$ is drawn from a common knowledge prior distribution μ. In all models we consider in this survey, the order of events for a game with n agents is as follows:

(1) The principal *commits* to a signaling scheme φ.
(2) Nature draws $\theta \sim \mu$.
(3) Signals $(\sigma_1, \ldots, \sigma_n) \sim \varphi(\theta)$ are drawn, and agent i learns σ_i.
(4) Agents select their strategies, and receive payoffs as determined by the game and the state of nature.

This survey adopts the perspective of the principal who has an objective in mind, and we focus on the optimization task faced by the principal in Step (1). A few notes on this general setup are in order. First, we note that signals are best thought of not as meaningful strings, but rather as abstract objects. Indeed, a rational agent interprets a signal by virtue of how it is used by the scheme φ, and therefore it has no intrinsic meaning beyond that. Second, it might seem unrealistic that the principal has the power to implement an arbitrarily informative signaling scheme. However, as pointed out in [Kamenica and Gentzkow 2011], this is without loss of generality: we can simply interpret the most informative signal the principal can access as the state of nature. Third, the reader might have noticed that we made no mention of information received by the agents which is out of the control of — and perhaps even unknown to — the principal. This is mostly for simplicity, and in fact some applications of the models we describe do involve agents who receive an exogenous signal, often referred to as the agent's *type*. Fourth — and this is related to the previous point — we restricted our attention to a one-step protocol of information revelation. In games where agents are privately informed, it might in fact be in the principal's interest to engage in a multi-round protocol where the principal and the agents exchange information piecemeal. The study of such protocols and settings is interesting in its own right, yet beyond the scope of this survey.

Last but not least, we justify what might at first appear as the most controversial assumption employed in most of the recent literature on information structure design. This is the *commitment assumption*: we assume that the principal has the power to credibly commit to the signaling scheme before realization of the state of nature. Without the power of commitment, the model becomes one of *cheap talk* (see [Crawford and Sobel 1982; Sobel 2010]). As is common in the recent literature on information structure design, we argue that the commitment assumption is not as unrealistic as it might first seem. One argument, mentioned in [Rayo and Segal 2010], is that commitment arises organically at equilibrium if the game is played repeatedly with a long horizon. This is because in such settings, the principal maximizes his long-term utility by establishing a reputation for credibility. A somewhat different argument, particularly suited for the algorithmic view of information structure design, is that any entity deploying an automated signaling scheme is likely to have a contractual service agreement with the agents, or otherwise have a vested interest in being perceived as a trusted authority. Trusting such an entity with the provision of information is not too unlike trusting an auctioneer to faithfully implement the rules of an auction, or trusting a certificate authority to properly issue digital certificates. In the case of signaling, commitment can involve publishing the source code of the signaling scheme used by the entity; agents can then verify the commitment over time through the use of statistical tests or audits. Additional justifications of the commitment assumption can be found in [Kamenica and Gentzkow 2011].

1.2 Structure of This Survey

We focus on three models, which to our knowledge capture or come close to capturing most recent work on information structure design. In Section 2 we consider the single-agent information structure design problem, also known as *Bayesian persuasion*. This is a special case of the next two models. In Section 3 we consider multiple agents, but a principal constrained to a public communication channel. The third model, considered in Section 4, affords the most power to the principal by permitting private communication between the principal and the individual agents. For all three of these models, we describe the mathematical setup, present structural characterizations (typically of a geometric nature) of the optimal information structure, discuss the state-of-the-art in algorithmic and complexity-theoretic work in that setting, and present open questions. We close this survey by briefly describing variations and extensions of these models in Section 5, and present some concluding thoughts in Section 6.

2. PERSUADING A SINGLE AGENT

Specializing information structure design to the case of a single agent yields the *Bayesian Persuasion* model proposed by [Kamenica and Gentzkow 2011], generalizing an earlier model by [Brocas and Carrillo 2007]. This is arguably the simplest model of information structure design, and the most applied. Indeed, the Bayesian persuasion model has been applied to a number of domains such as bilateral trade [Bergemann et al. 2015], advertising [Chakraborty and Harbaugh 2014], security games [Xu et al. 2015; Rabinovich et al. 2015], medical research [Kolotilin 2015], and financial regulation [Gick and Pausch 2012; Goldstein and Leitner 2013], just to mention a few. In addition to being interesting in its own right, the Bayesian persuasion model serves as a building block for more complex models of information structure design, and illustrates many of the basic principles which we will refer to in future sections of this survey.

2.1 The Model and Examples

In Bayesian persuasion, we adopt the perspective of a *sender* (the principal) looking to persuade a *receiver* (the single agent) to take an action which is desirable to the sender. There is a set A of actions, and the payoff of each action $a \in A$ to both the sender and the receiver is determined by the state of nature $\theta \in \Theta$ — we use $s(\theta, a)$ and $r(a, \theta)$ to denote those payoffs, respectively. We assume θ is drawn from a *common prior distribution* μ, and the sender must commit to a signaling scheme $\varphi : \Theta \to \Delta(\Sigma)$, where Σ denotes some set of signals and $\Delta(\Sigma)$ denotes the family of distributions over Σ. To illustrate this model, we look at a pair of examples.

EXAMPLE 2.1 (ADAPTED FROM [KAMENICA AND GENTZKOW 2011]). *Consider an academic adviser (the sender) who is writing a recommendation letter (the signal) for his graduating student to send to a company (the receiver), which in turn must decide whether or not to hire the student. The adviser gets utility 1 if his student is hired, and 0 otherwise. The state of nature determines the quality of the student, and hence the company's utility for hiring the student. Suppose that the student is excellent with probability $\frac{1}{3}$, and weak with probability $\frac{2}{3}$. Moreover, suppose that the company gets utility 1 for hiring an excellent student, utility -1 for hiring a weak student, and utility 0 for not hiring. Consider the following signaling schemes:*

—*No information: Given no additional information, the company maximizes its utility by not hiring. The adviser's expected utility is 0.*

—*Full information: Knowing the quality of the student, the company hires if and only if the student is excellent. The adviser's expected utility is $\frac{1}{3}$.*

—*The optimal (partially informative) scheme: The adviser recommends hiring when the student is excellent, and with probability just under 0.5 when the student is weak. Otherwise, the adviser recommends not hiring. The company maximizes its expected utility by following the recommendation, and the adviser's expected utility is just under $\frac{2}{3}$.*

EXAMPLE 2.2 (ADAPTED FROM [DUGHMI AND XU 2016]). *Example 2.1 can be generalized so that the receiver has many possible actions. The adviser has a number of graduating students, and the company must choose to hire one of them. The qualities of the different students are i.i.d.; specifically, each student is equally likely to be weak (W), a short-term achiever (S), and a long-term achiever (L). The company derives utility 0 from hiring a W student, utility $1 + \epsilon$ for hiring an S student, and utility 2 from hiring an L student. The adviser, on the other hand, is up for tenure soon and derives utility 1 if the company hires an S student and 0 otherwise. Consider the following signaling schemes:*

—*No information: All students appear identical to the company, which chooses arbitrarily. The hired student is of type S with probability $\frac{1}{3}$, and therefore the adviser's expected utility is $\frac{1}{3}$.*

—*Full information: Knowing the quality of all students, the company hires a student of type L whenever one is available. As the number of students grows large, the adviser's utility tends to 0.*

$$\begin{array}{lll} \text{maximize} & \sum_{\theta \in \Theta} \mu(\theta) \sum_{a \in A} \varphi(\theta, a) s(\theta, a) & \\ \text{subject to} & & \\ & \sum_{\theta} \mu(\theta) \varphi(\theta, a)(r(\theta, a) - r(\theta, a')) \geq 0, & \text{for } a, a' \in A. \\ & \sum_{a \in A} \varphi(\theta, a) = 1, & \text{for } \theta \in \Theta. \\ & \varphi(\theta, a) \geq 0, & \text{for } \theta \in \Theta, a \in A. \end{array}$$

Fig. 1. Linear Program for Optimal Bayesian Persuasion

—*The optimal (partially informative) scheme: When at least one student has type S, the adviser recommends one of the S students uniformly at random. Otherwise, he recommends a student uniformly at random. Using the fact that the company prefers a student of type S to an equal mixture of types W and L, a simple calculation using Bayes' rule reveals that the company maximizes its expected utility by following the adviser's recommendation. As the number of students grows large, the adviser's utility tends to* 1.

2.2 Characterization of the Optimal Scheme

The reader might notice that, in both Examples 2.1 and 2.2, the signals of the optimal scheme correspond to the different actions, and can be thought of as *recommending* an action to the receiver. Moreover, when the receiver is recommended an action a, this recommendation is *persuasive* in the sense that a maximizes the receiver's expected payoff with respect to the posterior distribution of the state of nature θ induced by the signal a. This is not a coincidence: as observed by [Kamenica and Gentzkow 2011], an argument similar to the *revelation principle* shows that every signaling scheme is equivalent to one which recommends an action subject to such persuasiveness — i.e., we may assume without loss that $\Sigma = A$.[1] This dimensionality-reduction holds not only in this single-agent setting, but also in multi-agent settings when the agents are sent private signals (more on this in Section 4).

Given this characterization, it is not hard to see that the sender's optimal signaling scheme — i.e., the scheme maximizing his expected utility — is the solution to a simple linear program (Figure 1). This LP has a variable $\varphi(\theta, a)$ for each state of nature θ and action a, corresponding to the conditional probability of recommending action a given state θ. Solving this LP is impractical unless the prior distribution μ is of small support and given explicitly, but it nevertheless serves as a useful structural characterization. The LP maximizes the sender's utility, in expectation over the joint distribution of θ and a, subject to persuasiveness. Another way to visualize the feasible region of this LP is instructive: the probability of signal a is $\Pr[a] = \sum_{\theta} \mu(\theta)\varphi(\theta, a)$, and the posterior distribution μ_a on states of nature induced by signal a is given by $\mu_a(\theta) = \frac{\mu(\theta)\varphi(\theta,a)}{\Pr[a]}$. Therefore, a feasible solution of the LP can be thought of as a distribution over posteriors — one per signal — whose expectation equals the prior μ. In other words, if the prior μ is represented by a point in the probability simplex $\Delta = \Delta(\Theta)$, then the signaling scheme corresponds to a way of writing μ as a convex combination of posterior

[1]Persuasiveness has been referred to as *obedience* or *incentive compatibility* in the prior literature on persuasion.

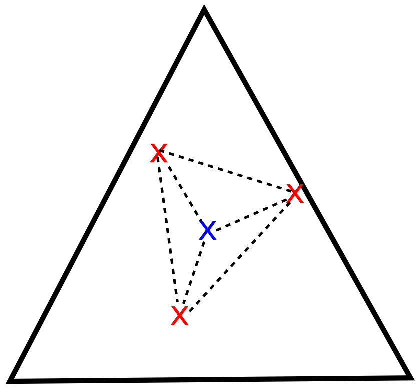

Fig. 2. A Signaling Scheme with Three Signals

distributions in Δ, as illustrated in Figure 2. Persuasiveness is equivalently stated as the constraint that action a is favored by a rational receiver facing the posterior distribution on θ induced by the signal a.

This geometric / LP interpretation reveals some structural properties of optimal signaling schemes. Each posterior distribution $\mu' \in \Delta$ is associated with a preferred action $a = a(\mu')$ for the receiver — this is the action maximizing the receiver's expected utility $\mathbf{E}_{\theta \sim \mu'} r(\theta, a)$.[2] This allows us to plot the sender's utility as a function $f : \Delta \to \mathbb{R}$ of the posterior: $f(\mu') = \mathbf{E}_{\theta \sim \mu'} s(\theta, a(\mu'))$. Let $\widehat{f}$ be the *concave closure*[3] of f; our geometric interpretation implies that the optimal signaling scheme achieves sender utility equal to $\widehat{f}(\mu)$ by optimally decomposing μ into posterior distributions. In other words, optimal signaling can be thought of as computing the concave closure of f (This was observed by [Kamenica and Gentzkow 2011]). This task is nontrivial only when f is neither convex nor concave. If f is concave then the optimal scheme reveals no information (i.e., recommends the receiver's ex-ante preferred action regardless of the state of nature). Whereas if f is convex the optimal scheme reveals all information (i.e., recommends the receiver's ex-post preferred action for each state of nature). For example, f is convex when sender and receiver utilities are always equal, and concave when they sum to zero. In general, this geometric interpretation yields a different bound on the number of signals via Caratheodory's theorem: the optimal scheme needs no more signals than the number of states of nature.

2.3 Independent Distributions and the Connection to Auctions

In mechanism design, the salient properties of optimal policies are often revealed when the underlying uncertainty admits a simple structure. The prime example of this is Myerson's characterization [Myerson 1981] of the optimal single-item auction: when bidder values are independent, the optimal auction maximizes what is known

[2]In case of ties, we break them in favor of the sender.
[3]The concave closure of a function f is the pointwise smallest concave function upperbounding f, obtained by taking the convex hull of the points beneath the plot of f.

as the *virtual welfare*, and when they are i.i.d. this optimal auction is furthermore a second-price auction with a reserve price. This characterization leads to an efficient algorithm implementing the optimal auction when bidders' distributions are independent and given explicitly. The analogous questions in the Bayesian Persuasion model are the following: *Is there an efficient algorithm for optimal persuasion when actions are independent or i.i.d.? Can this algorithm be captured by a simple "rule of thumb" à la virtual welfare maximization?*

The situation for persuasion turns out to be more nuanced than in the case of auctions. An efficient optimal algorithm exists in the case of i.i.d. actions (as in Example 2.2), but there is evidence that it might not exist in the case of independent non-identical actions. To be precise, when we say actions are independent we mean that $s(a) = s(\theta, a)$ and $s(a') = s(\theta, a')$ are independent random variables for different actions $a \neq a'$, and the same for $r(a) = r(\theta, a)$ and $r(a') = r(\theta, a')$. In this case, the distribution μ is fully specified by the marginal distribution of the pair $(s(a), r(a))$ for each action a. We assume that each action's marginal distribution has finite support, and refer to each element of this support as a *type*. We prove the following in [Dughmi and Xu 2016]; n denotes the number of actions and m denotes the number of types for each action.

Theorem 2.3 [Dughmi and Xu 2016]. *There is a polynomial-time (in n and m) algorithm implementing the optimal scheme for Bayesian persuasion with i.i.d. actions.*

Theorem 2.4 [Dughmi and Xu 2016]. *Unless $P = \#P$, there is no polynomial-time (in n and m) algorithm for evaluating the sender's optimal utility in Bayesian persuasion with independent actions.*

Theorem 2.3 relies on a connection to auction theory, and in particular to Border's characterization of reduced-form allocation rules for a single-item auction. The analogy to a single-item auction is as follows: each action corresponds to a bidder, the types of an action correspond to the types of bidders, and recommending an action corresponds to selecting the winning bidder. In this analogy, the joint distribution of recommended action and its type is simply what is known as the *reduced-form allocation rule* of a single-item auction. The space of reduced forms has been characterized by [Border 1991; 2007], and has since been observed to be a polytope admitting an efficient separation oracle [Cai et al. 2012; Alaei et al. 2012]. Therefore, computing the optimal scheme reduces to linear optimization over this polytope, augmented with persuasiveness constraints instead of auction incentive-compatibility constraints. We show that in the i.i.d. setting, persuasiveness constraints are linear in the reduced form, and the resulting linear program can be solved efficiently. This leads to the reduced form allocation rule associated with the optimal signaling scheme, and the optimal scheme can be implemented from that using Bayes' rule. Notably, whereas this algorithm is efficient, it arguably lacks the simplicity of Myerson's optimal auction — the difference is due to replacing the auction incentive-compatibility constraints with persuasiveness constraints.

Theorem 2.4 shows that, unlike in the case of auctions, the tractability of persuasion in the i.i.d. setting does not appear to directly extend to the independent setting. The culprit yet again are the persuasiveness constraints: unlike in the i.i.d.

setting, these can no longer be expressed as linear constraints in the reduced form. Moreover, our result shows that no generalization of the reduced form works either, ruling out a *generalized Border's theorem* for persuasion, in the sense of [Gopalan et al. 2015]. That being said, we note that this result rules out exactly computing the principal's optimal utility from signaling in polynomial time, yet does not rule out efficiently sampling the output of the optimal signaling scheme "on the fly", input by input.

2.4 General Black-Box Distributions

The previous subsection illustrates how simplifying the input model and using computation as a lens can lead to structural insights into optimal schemes. Another approach, common when computer scientists seek unifying and simplifying algorithmic and structural results, is the following: fix an input model which is as general as possible, and design an algorithm which succeeds regardless of the details of the setting. In Bayesian persuasion, the most general model permits an arbitrary joint distribution of sender and receiver payoffs from the various actions, allowing arbitrary correlations between the payoff-relevant variables. To be precise, there are $2n$ payoff-relevant random variables in total, where n is the number of actions: each action is associated with a payoff to both the sender and the receiver. In [Dughmi and Xu 2016], we assume that this joint distribution can be sampled from, and moreover that all the variables lie in a bounded interval (without loss of generality $[0, 1]$), and prove the following.

THEOREM 2.5 [DUGHMI AND XU 2016]. *For general payoff distributions presented as a sampling black box, an ϵ-optimal and ϵ-persuasive scheme (in the additive sense) for Bayesian persuasion can be implemented in time polynomial in n and $\frac{1}{\epsilon}$. Moreover, this bi-criteria loss is inevitable in the worst case: an optimal scheme must be $\Omega(1)$ far from being persuasive, and a persuasive scheme must be $\Omega(1)$ far from optimality.*

The positive statement in Theorem 2.5 concerns a simple scheme based on Monte-Carlo sampling: When our signaling scheme is queried with a state of nature $\theta \in \Theta$, it additionally samples polynomially many times from the prior μ to get a set $S \subseteq \Theta$, solves a relaxed variant of the LP in Figure 1 on the empirical distribution $S \cup \{\theta\}$, and produces a recommendation as suggested by the LP for θ.

Reflecting on Theorem 2.5 and the associated scheme, we can conclude that Bayesian persuasion admits an efficient, approximately-optimal, and approximately-persuasive scheme in arbitrarily general settings. Moreover, this scheme is simple: additional samples S are procured in order to place the query θ "in context" of the prior distribution μ, and the algorithm "pretends" that the prior distribution is the uniform distribution on $S \cup \{\theta\}$. Naturally, this succeeds due to convergence of the LP solution to the optimal solution. The LP in Figure 1 is modified by relaxing the persuasiveness constraints, as it so happens that this prevents "overfitting" to the sample. The negative statement in Theorem 2.5 shows that this relaxation is inevitable for information-theoretic reasons: one can construct examples where the sender — having only imprecise sampling knowledge of the prior distribution — can not be certain whether his preferred recommendation is persuasive to a receiver who has full knowledge of the prior.

2.5 Future Work and Open Questions

We now mention three open questions of an algorithmic flavor pertaining to the Bayesian persuasion model.

Open Question 2.6. *Consider Bayesian persuasion with n independent actions, each having a marginal distribution supported on m types. Is there a polynomial-time (in n and m) implementation of the optimal signaling scheme? If so, what does this algorithm reveal about the structure of the optimal policy?*

Recall that Theorem 2.4 rules out efficiently computing the optimal expected sender utility, yet does not preclude sampling $\varphi^*(\theta)$ for each input θ, where φ^* is an optimal scheme. This is a subtle point, but is not unprecedented: [Gopalan et al. 2015] exhibit simple auction settings in which the optimal revenue of the auctioneer is $\#P$ hard to compute, and yet Myerson's optimal auction can be efficiently implemented input-by-input. Theorem 2.4 implies that an optimal signaling scheme cannot be efficiently computed using linear-programming approaches à la Border's theorem. Therefore, if a similar phenomenon occurs here, the algorithm implementing the optimal signaling scheme would likely reveal some important structure of optimal persuasion, à la Myerson's famous characterization of optimal auctions as virtual-welfare maximizers [Myerson 1981]. This leads right into our next open question.

Open Question 2.7. *Can optimal Bayesian persuasion be described by a simple "rule of thumb" policy, à la virtual welfare maximization from auction theory?*

The results of [Dughmi and Xu 2016] do not answer this question, even in the simplest case of i.i.d. actions. Indeed, the result in Theorem 2.3 invokes Border's theorem to compute the entire reduced form for the optimal scheme, rather than provide a simple input-by-input rule such as virtual welfare maximization.

Our final question concerns moving beyond an explicitly-given list of actions. In a number of natural applications of persuasion, the receiver's action is naturally multidimensional — say a path in a network, a point in space, or an allocation of resources among different projects. In such settings, the actions lie in a vector space, and the receiver faces an optimization problem — say, encoded as a linear or convex program — when choosing their optimal action. When the state of nature determines the objective function of both the sender and the receiver, can we solve for an approximately-optimal signaling scheme in time polynomial in the natural parameters of the problem? Say, in the number of variables and constraints of the linear program rather than the number of its vertices (the possible actions)?

Open Question 2.8. *Consider Bayesian persuasion with multidimensional actions. In what settings can an optimal or near-optimal signaling scheme be computed in time polynomial in the dimensionality of the receiver's optimization problem?*

3. MULTIPLE AGENTS: PUBLIC SIGNAL

Information structure design can get much more intricate when multiple players are involved, particularly if they have heterogeneous beliefs, or if a scheme induces heterogeneous beliefs through revealing different information to different players. In this section, we examine a model which simplifies away such considerations:

	Coop	Defect
Coop	$-1+\theta$ $-1+\theta$	0 $-5+\theta$
Defect	$-5+\theta$ 0	-4 -4

Fig. 3. An incomplete information variant of the prisoners' dilemma.

all players (including our principal) share the same common prior on the state of nature, and we constrain our principal to a public communication channel — i.e., all players in the game receive the same information. This *public signal* model underlies much of the work on multi-agent information structure design, such as [Emek et al. 2012; Bro Miltersen and Sheffet 2012; Guo and Deligkas 2013; Dughmi et al. 2014] in the context of auctions, and [Alonso and Cmara 2016a; 2016b] in the context of voting.

3.1 The Model and Examples

An n-player game of incomplete information specifies a set A_i of actions for each player i, a set Θ of states of nature, and a payoff function $\mathcal{G} : \Theta \times A_1 \times \ldots \times A_n \to \mathbb{R}^n$, where $\mathcal{G}_i(\theta, a_1, \ldots, a_n)$ is player i's payoff when the state of nature is θ and each player j plays action a_j. The game may be represented explicitly via its normal form, or via some implicit representation permitting evaluation of the function $\mathcal{G}$.

We assume that the state of nature θ is distributed according to a *common prior distribution* $\mu \in \Delta(\Theta)$. A principal must commit to a signaling scheme $\varphi : \Theta \to \Delta(\Sigma)$, where Σ is some set of signals. We interpret the (random) output $\sigma \sim \varphi(\theta)$ as a public signal which is received by all players in the game, say through a public communication channel. The payoff function $\mathcal{G}$, the prior μ, and the signaling scheme φ then define a *Bayesian Game* in the classical game-theoretic sense. We naturally assume that players react by playing a Bayesian Nash Equilibrium in this game, possibly according to some domain-specific equilibrium selection rule in the case of multiple equilibria.

We adopt the perspective of the principal, looking to maximize some function in expectation over the realized state of nature and action profiles. At its most general, the principal's utility function is of the form $\mathcal{F} : \Theta \times A_1 \times \ldots A_n \to \mathbb{R}$. We present some examples below to make this model concrete.

Example 3.1 [Dughmi 2014]. *An incomplete information variant of the classical prisoners' dilemma is shown in Figure 3. The game's payoffs are parametrized by a state of nature θ. When $\theta = 0$, this is the traditional prisoners' dilemma in which cooperation is socially optimal, yet the unique Nash equilibrium is the one where both players defect, making both worse off. Assume, however, that θ is uniformly distributed in $[-3, 3]$, and assume the principal wishes to maximize the social welfare. Consider the following public signaling schemes:*

—*No information: The players, being risk neutral, play as if θ equals its expectation of 0. Defection dominates and the social welfare is -8.*

—*Full information: Defection dominates when $\theta < 1$ yielding welfare -8, and*

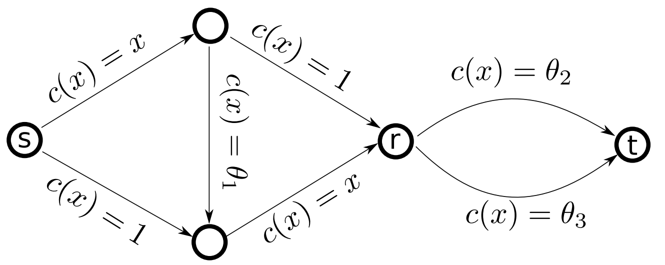

Fig. 4. An incomplete-information Routing Game

cooperation dominates when $\theta \geq 1$ yielding welfare $2\theta - 2$. The expected social welfare is $-\frac{14}{3}$.

—*The optimal (partially informative) scheme: When $\theta > -1$ signal* `High`*, and otherwise signal* `Low`*. Given signal* `High`*, which is output with probability $\frac{2}{3}$, both players' posterior expectation of θ is 1, and therefore cooperation dominates. Defection dominates when the signal is* `Low`*. The expected social welfare is $-\frac{8}{3}$.*

EXAMPLE 3.2. *In a* non-atomic selfish routing game *(see [Nisan et al. 2007]), there is continuum of selfish agents looking to travel from a source s to a sink t in a directed network, and each edge of the network is labeled with a congestion function measuring the cost incurred by each agent as a function of the total fraction of flow using that edge. Consider the incomplete-information routing game depicted in Figure 4, in which the congestion functions are determined by a state of nature $\theta = (\theta_1, \theta_2, \theta_3)$. This network consists of a variant of the* Braess paradox *network (see [Nisan et al. 2007]), followed by a pair of parallel edges feeding into the sink.*

Suppose that the three components of θ are i.i.d. random variables, each uniformly distributed on $[0, 1]$. Suppose also that the principal wishes to minimize the social cost, i.e. the average congestion experienced by the agents, at equilibrium, assuming agents are risk neutral. Consider the first portion of the journey, from s to r; As in Braess's paradox, the equilibrium routing from s to r is suboptimal when agents believe that the expectation of θ_1 is less than 0.5, and optimal otherwise. Therefore, it behooves the principal to reveal no information about θ_1 — this can be thought of as an informational Braess's paradox. *In contrast, for the final hop of the journey from r to t the principal optimizes the social cost by revealing θ_2 and θ_3, as this assists all agents in choosing the lower congestion edge.*

To summarize, the optimal scheme reveals some components of the state of nature and withholds others.

EXAMPLE 3.3. *In a* probabilistic single-item auction*, an item with a priori unknown attributes is being auctioned to a set of bidders. This arises in online advertising, where advertisers must bid on an impression, and this impression is associated with a web user drawn from a population of web users. [Emek et al. 2012] and [Bro Miltersen and Sheffet 2012] study optimal public signaling policies by a revenue-maximizing auctioneer in this setting, assuming the second-price auction format is fixed. In some settings — such as when bidders have similar preferences and the market is highly competitive — the optimal policy reveals all information* *about the item for sale. In other settings — such as when bidders have idiosyncratic preferences and markets are "thin" — withholding much of the information about the item can increase competition and drive up prices. In general, optimal policies reveal partial information. We refer the reader to [Emek et al. 2012; Bro Miltersen and Sheffet 2012] for a more detailed discussion.*

3.2 Characterization of the Optimal Scheme

As in the case of a single agent, we can identify a signaling scheme $\varphi : \Theta \to \Delta(\Sigma)$ with a way of writing the prior distribution μ as a convex combination of posterior distributions $\{\mu_\sigma : \sigma \in \Sigma\}$ (See Figure 2), where μ_σ is the posterior distribution on the state of nature given the signal σ. Unlike in the case of a single agent, however, we can no longer identify signals with actions. Indeed, each signal $\sigma \in \Sigma$ induces a subgame in which the common prior on the state of nature is μ_σ, and players might play a mixed Nash equilibrium in this subgame.

Fixing an equilibrium selection rule and denoting $\Delta = \Delta(\Theta)$, like in the single-agent case we get an objective function $f : \Delta \to \mathbb{R}$ mapping posterior distributions to the principal's expected utility for the resulting equilibrium. Like in the single-agent case, we can therefore also interpret the optimal scheme as evaluating the concave closure $\widehat{f}$ of f by optimally decomposing the prior μ into posterior distributions. Whereas we can no longer bound the number of signals by the number of actions, the bound from Caratheodory's theorem still holds: the optimal scheme needs no more signals than the number of states of nature.

3.3 Negative Results

Recall that in the single-agent setting of Section 2, a simple linear program expresses the optimal signaling task. Moreover, this LP is of size linear in the normal form of that game — i.e., linear in the number of (state of nature, action) pairs. This simplicity is largely a consequence of the dimensionality-reduction property in the single-agent case, and underlies the positive algorithmic and structural results outlined in Section 2.

It is natural to ask how quickly this structure deteriorates as we move beyond a single agent. The answer for the public signal model: very quickly. A series of works [Dughmi 2014; Bhaskar et al. 2016; Rubinstein 2015] examines signaling in 2-player zero-sum games, and culminates in the following.

THEOREM 3.4 [BHASKAR ET AL. 2016; RUBINSTEIN 2015]. *Consider a 2- player Bayesian zero-sum game with payoffs in* $[0, 1]$*, presented via its normal form, and a principal interested in maximizing the utility of one of the players. The principal's signaling problem is NP-hard [Bhaskar et al. 2016], and moreover does not admit an additive PTAS*[4] *assuming either the* planted clique conjecture *[Bhaskar et al. 2016] or the* exponential time hypothesis *[Rubinstein 2015].*

These impossibility results have significant bite: they hold in a setting where equilibrium selection and computation are a non-issue — all equilibria of a 2-player zero-sum game are equivalent to the efficiently-computable minimax equilibrium.

[4] A Polynomial-Time Approximation Scheme (PTAS) is an algorithm which, given any constant $\epsilon > 0$, computes an ϵ-approximately optimal solution in time polynomial in the size of the instance.

Moreover, maximizing one player's utility can be shown to be no harder than other natural choices of objective function, such as the social welfare (weighted or unweighted).[5] An arguably reasonable reading of these results is that *a simple and general characterization of optimal public signaling is unlikely to exist even in the simplest of multiagent settings.*

We mention two other impossibility results in specific game domains which reinforce this message. [Emek et al. 2012] show that revenue-maximizing public signaling in the second price auction (à la example 3.3) is NP-hard, and [Bhaskar et al. 2016] show that welfare-maximizing public signaling in selfish routing with linear congestion functions (à la example 3.2) is NP-hard to approximate to within a multiplicative factor better than $\frac{4}{3}$ — this is the price of anarchy in this setting, and is trivially achievable by the fully-informative scheme.

3.4 Positive Results: Exploiting "Smoothness"

The impossibility results stated in Theorem 3.4 involve constructing games where a near-optimal signaling scheme must reveal much — but not all — of the information contained in the state of nature. For example, the reductions in Theorem 3.4 feature a state of nature which is a random node in an n-node graph, and a near-optimal scheme must essentially partition the nodes into equivalence classes of size roughly $\sqrt{n}$, in effect revealing roughly half of the bits of information. Informally, this induces a combinatorial search problem which searches for the "right" half of the bits to reveal. It is not entirely surprising, therefore, that this problem can be intractable.

One might wonder if there are natural classes of games in which we can get away with revealing much less information. This would simplify the search problem, making it more computationally tractable. This idea is explored by [Cheng et al. 2015], who identify two "smoothness" properties which seem to govern the complexity of near-optimal signaling schemes. Suppose that each state of nature $\theta \in \Theta$ is a vector of N "relevant parameters" in a bounded interval — for example, in a probabilistic single-item auction N may be the number of different bidder types, and $\theta_t \in [0, 1]$ may be the item's value for bidders of type t. Moreover, suppose that the equilibrium of the game, and therefore the principal's utility, depend only on the posterior expectation of each of the relevant parameters. This is the case in the auction setting (Example 3.3), where risk-neutral bidders bid their expected value for the (random) item being sold. Finally, suppose that the game is "smooth" in two respects:

- α-Lipschitz continuity in L^∞: If each relevant parameter changes by at most ϵ, then the principal's utility does not decrease by more than $\alpha\epsilon$.
- β-Noise stability: Suppose that an adversary corrupts (i.e. changes arbitrarily) a random subset R of the relevant parameters, where no individual parameter is in R with probability more than ϵ. The principal's utility does not decrease by more than $\beta\epsilon$.

[5] It is clear how this is a special case of weighted welfare. Moreover, multiplying player 2's utility by a small constant approximately reduces the problem of maximizing player 1's utility to maximizing the unweighted social welfare.

Theorem 3.5 [Cheng et al. 2015]. *Suppose that a game is α-Lipschitz and β-noise stable in its relevant parameters, and let $\epsilon > 0$ be an approximation parameter. There is an ϵ-approximately optimal signaling scheme (in the additive sense) where each posterior belief is a uniform distribution over $(\frac{\alpha}{\epsilon})^2 \log(\frac{\beta}{\epsilon})$ states of nature. When α and β are constants, such a scheme can be computed in time polynomial in the number of states of nature and the number of relevant parameters, yielding an additive PTAS.*

This theorem implies a PTAS for revenue-maximizing signaling in the probabilistic second-price auction, since the second-price auction is 1-Lipschitz and 2-stable. For the former, observe that changing bids by no more than ϵ can only change the second price by at most ϵ. For the latter, if each bid is corrupted to an arbitrary value with probability at most ϵ, then with probability at least $1 - 2\epsilon$ the top two bids are untouched and the revenue does not decrease.

These two conditions, Lipschitz continuity and noise stability, imply that "small" posterior beliefs suffice for a near-optimal public signaling scheme. The proof of this portion of Theorem 3.5 proceeds by decomposing each posterior belief μ_σ of the optimal scheme into "small" posterior beliefs by sampling; i.e., μ_σ is written as the average of empirical distributions sampled from it. Sampling from a distribution over states of nature leads to (a) high-probability small errors in the relevant parameters, the effect of which is bounded using Lipschitz continuity of the objective; and (b) low-probability large errors, the effect of which is bounded using noise stability of the objective.

Once we can restrict attention to small posteriors, the computational task becomes tractable. Specifically, computing a near-optimal scheme reduces to a brute force search over small posterior distributions, using linear programming in order to assemble the convex decomposition of the prior distribution.

In addition to probabilistic second-price auctions, these ideas have been applied in [Cheng et al. 2015] to signaling in the voting setting of [Alonso and Cmara 2016a] yielding an approximation scheme, and to derive a quasipolynomial-time approximation scheme for signaling in normal-form games.

3.5 Future Work and Open Questions

We mention one direction for future work, concerning the extent to which the restriction to public signals is binding.

Open Question 3.6. *In what classes of games and objectives is a public signaling policy optimal or near optimal, when evaluated against the optimal policy which can send private signals?*

In other words, when does moving to private signals — which we discuss in Section 4 — not buy the principal additional power? An answer to this question in the special class of games with no inter-agent externalities and binary actions is provided by [Arieli and Babichenko 2016] (we discuss their model in detail in Section 4.3).

4. MULTIPLE AGENTS: PRIVATE SIGNALS

We now examine a model which affords the principal the most power, allowing him to tailor his signal to individual players. As in Sections 2 and 3, we still assume that the principal and all the agents share the same common prior on the state of nature. However, since the principal reveals different information to different agents, the agents' posterior beliefs will differ.[6] Though information structure design with private signals has not been very thoroughly explored, particularly algorithmically, most recent applications of private signaling fall under the model we outline in this section. Specifically, we mention the work of [Taneva 2015] who characterizes optimal information structures in two-agent two-action games, [Bardhi and Guo 2016] who study persuading voters in a unanimity election, and [Arieli and Babichenko 2016; Babichenko and Barman 2016] who study persuading multiple agents facing a binary action with no inter-agent externalities.

4.1 The Model and Examples

As in Section 3, we have an n-player game of incomplete information $\mathcal{G} : \Theta \times A_1 \times \ldots \times A_n \to \mathbb{R}^n$, a common prior distribution $\mu \in \Delta(\Theta)$ over states of nature, and an objective function $\mathcal{F} : \Theta \times A_1 \times \ldots A_n \to \mathbb{R}$. We again adopt the perspective of a principal who designs and commits to a signaling scheme with the goal of maximizing $\mathcal{F}$ in expectation, but unlike Section 3 this signaling scheme is of the form $\varphi : \Theta \to \Delta(\Sigma_1 \times \ldots \times \Sigma_n)$, where Σ_i is a set of signals intended for player i. The output of φ is a random signal profile $(\sigma_1, \ldots, \sigma_n)$, where σ_i is sent to payer i via a private channel. Together, $\mathcal{G}$, μ and φ define a Bayesian game.

We present two examples to illustrate the model, and to show how private signaling affords more power to the principal than does public signaling.

Example 4.1 (Adapted from [Arieli and Babichenko 2016]). *As in Example 2.1, consider an academic adviser who is writing a recommendation letter for his student. However, now the student has applied to two fellowship programs, each of which must decide whether or not to award the student a fellowship funding part of his graduate education. Suppose that the student can accept one or both fellowship awards. The adviser, who has enough grant funding for most (but not all) of his student's education, gets utility* 1 *if his student is awarded at least one fellowship, and* 0 *otherwise. As in Example 2.1, the student is excellent with probability* $1/3$ *and weak with probability* $\frac{2}{3}$*, and a fellowship program gets utility* 1 *from awarding an excellent student,* -1 *from awarding a weak student, and* 0 *from not awarding the student. Naturally, a fellowship program makes an award if and only it believes its expected utility for doing so is nonnegative.*

Consider the following signaling schemes:

—*No Information: Neither program makes the award, and the adviser's utility is* 0.

—*Full information: Both programs make the award if the student excellent, and neither makes the award if the student is weak. The adviser's expected utility is* $1/3$.

—*Optimal public scheme: If the student is excellent, the adviser publicly signals "award". If the student is weak, the adviser publicly signals "award" or "don't award" with equal probability. This scheme is the same as the optimal scheme for the single-receiver version of this example (Example 2.1), extended to both*

[6]Equally importantly, their higher order posterior beliefs regarding each other are different and nontrivial.

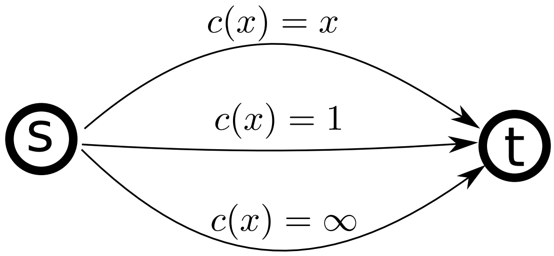

Fig. 5. Adapted Pigou Example

receivers via publicizing the recommendation letter. Therefore, both programs are simultaneously persuaded to award the student the fellowship with probability $2/3$*, and neither makes the award with probability* $1/3$*. The adviser's expected utility is* $2/3$*.*

—*Optimal private scheme: If the student is excellent, the adviser recommends "award" to both fellowship programs. If the student is weak, the adviser recommends "award" to one fellowship program chosen uniformly at random, and recommends "don't award" to the other. Notice that, from an individual program's perspective this is the same as the previous scheme, the difference being that the recommendations are anticorrelated when the student is weak. The result is that both fellowship programs make the award when the student is excellent, and exactly one of the programs makes the award when the student is weak. This yields utility* 1 *for the adviser.*

Example 4.2 [Cheng and Xu 2016]. *This example concerns non-atomic selfish routing, as in Example 3.2. Consider the adaptation of the Pigou routing network (see [Nisan et al. 2007]) depicted in Figure 5. The optimal routing evenly splits the agents between the* x *and* 1 *edges leading to a social cost of* $\frac{3}{4}$*, whereas the full-information equilibrium routing sends all agents along the* x *edge leading to a social cost of* 1*. Suppose that nature applies a cyclic permutation to the tuple of congestion functions* $(x, 1, \infty)$ *— i.e. there are three states of nature. The principal can influence the routing by revealing information about the state of nature. Consider the following signaling schemes.*

—*No information: The average congestion is* ∞*, since one third of the agents will end up using the* ∞ *edge.*

—*Full information: The agents avoid the* ∞ *edge, and arrive at an equilibrium routing for the remaining Pigou network. At equilibrium, all travelers use the* x *edge, leading to a social cost of* 1*.*

—*Optimal public scheme: Though it takes proof, it can be shown that no public signaling scheme can do better than revealing full information. Intuitively, the principal's signal must identify at least one of the non-*∞ *edges with certainty, lest any of the agents use the* ∞ *edge.*[7] *If both non-*∞ *edges are identified, then this fixes the cyclic permutation and is equivalent to revealing full information.*

[7]By this we mean that the principal's signal must allow agents to conclude with certainty that

If, however, exactly one of the non-∞ edges is identified, all agents will use that edge in order to avoid the ∞ edge, leading to a social cost of 1 *as well.*

—*Optimal private scheme: The principal identifies one of the non-∞ edges to half the agents, and the other non-∞ edge to the other half. In other words, the principal privately recommends the x edge to a random half of the agents and privately recommends the* 1 *edge to the other half, without revealing whether the recommended edge has congestion function x or* 1*. Each agent is persuaded to follow the recommendation, since deviating from the recommendation lands them on the ∞ edge with probability $\frac{1}{2}$. This is the optimal routing for this network, and yields a social cost of $\frac{3}{4}$.*

4.2 Characterization of the Optimal Scheme

In both examples 4.1 and 4.2, the optimal private scheme recommends an action to each agent, and correlates these recommendations in a manner that optimizes the principal's objective. As in the single agent case (Section 2), this is not a coincidence: a revelation-principle-style argument reveals any signaling scheme is equivalent[8] to one which makes *persuasive* recommendations. In this multi-agent context, we say a recommendation scheme is persuasive if the agent maximizes his expected utility by always choosing the recommended action, assuming all other agents follow the recommendation. Equivalently, a scheme φ is persuasive if the strategy profile where each agent follows the recommendation forms a Bayes-Nash equilibrium of the Bayesian game $(\mathcal{G}, \mu, \varphi)$.

The above discussion might remind the astute reader of the *correlated equilibrium.* In fact, the joint distribution of action profile and state of nature induced by a signaling scheme at equilibrium forms what [Bergemann and Morris 2016] call a *Bayes Correlated Equilibrium (BCE).* Conversely, every BCE is induced by some persuasive scheme. Given a game of incomplete information $\mathcal{G}$ and a prior distribution μ on states of nature, one way to think of a BCE is as follows: It is a correlated equilibrium of $\mathcal{G}$ when we interpret nature as a player in the game,[9] endow the nature player with a trivial (i.e., constant everywhere) payoff function, and constrain the nature player's marginal strategy to be equal to μ. Therefore, like in the case of the correlated equilibrium, the space of BCEs can be expressed by a set of linear inequalities, and optimization over BCEs — equivalently, over signaling schemes — can be written as a linear program.

The LP for optimizing over BCEs is shown in Figure 6. Here, $\boldsymbol{a}$ ranges over action profiles, $\boldsymbol{a_{-i}}$ ranges over action profiles of players other than i, and θ ranges over states of nature. This LP generalizes the single-agent persuasion LP in Figure 1, modulo a simple change of variables. More interestingly, this LP is the same as the LP for optimizing over correlated equilibria (see e.g. [Papadimitriou and Roughgarden 2008]) with the exception of two differences: (a) the nature player has no incentive constraint, (b) the nature player's marginal distribution is constrained to

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a particular edge does not have congestion function ∞; the signal may, however, provide no information on whether that same edge has congestion function x or 1.

[8]We say two signaling schemes are equivalent if they induce the same joint distribution of action profile and state of nature at equilibrium.

[9]Under this interpretation, $\mathcal{G}$ becomes a game of *complete* information

$$
\begin{array}{lll}
\max & \sum_{\theta,\boldsymbol{a}} x(\theta,\boldsymbol{a})\mathcal{F}(\theta,\boldsymbol{a}) & \\
\text{s.t.} & & \\
& \sum_{\theta,\boldsymbol{a_{-i}}} x(\theta,a_i,\boldsymbol{a}_{-i})[\mathcal{G}_i(\theta,a_i,\boldsymbol{a_{-i}}) - \mathcal{G}_i(\theta,a_i',\boldsymbol{a_{-i}})] \geq 0, & \text{for } i \in [n], a_i \in A_i, a_i' \in A_i. \\
& \sum_{\boldsymbol{a}} x(\theta,\boldsymbol{a}) = \mu(\theta), & \text{for } \theta \in \Theta. \\
& x(\theta,\boldsymbol{a}) \geq 0, & \text{for } \theta \in \Theta, \boldsymbol{a} \in A_1 \times \ldots \times A_n.
\end{array}
$$

Fig. 6. Linear Program for Finding the Optimal Bayes Correlated Equilibrium

equal the prior distribution. A solution x to this LP (a BCE) corresponds to the persuasive private signaling scheme which, given a state of nature θ, recommends action profile $\boldsymbol{a}$ with probability $\frac{x(\theta,\boldsymbol{a})}{\mu(\theta)}$.

This characterization in terms of the Bayes correlated equilibrium exposes the dual role of an information structure: (1) informing players, which in effect allows them to correlate their actions with the state of nature, and (2) coordinating players by serving as a correlating device (as in the correlated equilibrium).

4.3 The Case of No Externalities and Binary Actions

To our knowledge, there has not been much algorithmic work on the use of private signals to influence agent behavior in multiagent settings. The recent exception is the *private Bayesian persuasion* model introduced by [Arieli and Babichenko 2016] and explored via the computational lens by [Babichenko and Barman 2016]. This model restricts information structure design to games with two simplifying features: (1) One agent's action does not impose an externality (positive or negative) on the other agents, and (2) each agent has a binary choice of action. The no-externality assumption implies that each agent's utility can be written as a function of just the state of nature θ and that particular agent's action, without any dependence on the actions of others. The principal's objective, on the other hand, may depend arbitrarily on the joint action profile of all the agents (as well as the state of nature). Since each agent's action is binary, without loss $A_i = \{0, 1\}$, the principal's objective can be equivalently described by a set function f, where $f(S)$ is the principal's utility if S is the set of agents taking action 1. In most natural examples, action 1 corresponds to adoption of an product or opinion, and 0 corresponds to non-adoption.

This model is illustrated by Example 4.1. The no-externality assumption implies that the principal faces n different Bayesian persuasion problems, one per agent, each with a binary action space. However, solving these problems separately can be suboptimal due to the non-modular dependence of the principal's objective on the agents' actions. In fact, the signaling problem can often be thought of as how to optimally correlate the solutions to the n (single-agent) Bayesian persuasion problems in order the maximize the principal's expected utility. This is indeed the case for Example 4.1: as a result of the adviser's submodular objective function, the optimal scheme anti-correlates the fellowship programs' actions as much as possible, subject to maximizing the marginal probability of a fellowship award in both cases.

More generally, the results of [Arieli and Babichenko 2016] demonstrate that a submodular objective function encourages anticorrelating the agents' recommendations (as in Example 4.1), and a supermodular objective function encourages correlation.[10] As an example of the latter, consider changing the adviser's utility function in Example 4.1 so that the adviser gets utility 1 if both fellowships are awarded and 0 otherwise; some thought reveals that the public scheme which persuades both fellowship programs to simultaneously award with probability $\frac{2}{3}$ becomes optimal. If the principal's objective function is modular (a.k.a. linear) in the set of persuaded agents, such as if the adviser's utility equals the number of fellowships awarded, then it is optimal to solve the n Bayesian persuasion problems separately — i.e., correlation in agents' actions does not affect the principal's utility.

[Babichenko and Barman 2016] go on to examine computing or approximating the principal's optimal signaling policy. In the case of a binary state of nature, they show that the problem of optimally correlating agents' actions is equivalent to that of computing the *concave closure* of the principal's objective, viewed as a set function. This makes a lot of sense, since the concave closure of a set function f maps a profile of marginal probabilities — in our case a probability of persuading each agent to take action 1 — to the maximum expected value of f for a random set S respecting those marginals — in our case, S is the random set of "persuaded" agents, and the optimal choice correlates or anticorrelates agents' actions in order to maximize the expectation of $f(S)$. It is known that the concave closure of a supermodular function can be computed efficiently (see e.g. [Dughmi 2009]), and it is shown in [Babichenko and Barman 2016] that the concave closure of a submodular function can be efficiently approximated to within a factor of $\frac{e}{e-1}$, and this is the best possible assuming $P \neq NP$. This connection leads to the following Theorem.

Theorem 4.3 [Babichenko and Barman 2016]. *Consider the private Bayesian persuasion model with a binary action space and binary state of nature. If the principal's objective function is supermodular (or modular), then there is a polynomial-time optimal signaling scheme. If the principal's objective function is submodular, then there is a polynomial-time $\frac{e}{e-1}$-approximately optimal signaling scheme, and this is the best possible assuming $P \neq NP$.*

4.4 Future Work and Open Questions

With the exception of the work described in Section 4.3, the computational aspects of information structure design with private signals remain largely unexplored territory. For one, [Babichenko and Barman 2016] leave open the natural generalization of their algorithmic questions to a non-binary state of nature. More generally, it remains to explore the algorithmics of private signaling in games with inter-agent externalities and non-binary action spaces.

Open Question 4.4. *When does private signaling admit simple and computationally efficient schemes which are optimal or near optimal?*

Since the optimal private signaling scheme is the solution to a large linear program (Figure 6), this question is most interesting when the game is described via some succinct representation, or given implicitly as an oracle for evaluating the payoff

[10] Though these characterization results are only formally stated for a binary state of nature, the conceptual message seems to hold more generally.

function. One class of games which seems to capture the challenges involved is the class of non-atomic routing games of incomplete information, illustrated in Example 4.2.

5. ADDITIONAL MODELS AND EXTENSIONS

This survey attempted to summarize the dominant models in information structure design, particularly as it relates to recent work which explores computational aspects of the question. However, we inevitably cannot capture the full breadth of work in this area. This section briefly describes a selection of models beyond the three we focus on in this survey.

Motivated by specific applications, a number of works in the computer science community have explored variants and extensions of the basic models from a computational perspective. A pair of papers consider optimal signaling subject to a constraint on the amount of communication (i.e., the number of signals): [Dughmi et al. 2014] consider public signaling subject to a communication constraint in an auction context, and [Dughmi et al. 2016] study Bayesian persuasion subject to a communication constraint both in general and in the context of bilateral trade. Motivated by recommender systems on the Internet, [Kremer et al. 2014; Mansour et al. 2015] consider a multi-armed bandit setting and a principal seeking to persuade a sequence of agents to balance exploration with exploitation over time. This can be viewed as a repeated game which combines information revelation with information acquisition, and the principal's interaction with each individual agent can be viewed as a Bayesian persuasion problem. Motivated by applications to security games, [Conitzer and Korzhyk 2011; Xu et al. 2016] consider a Stackelberg setting where the leader commits to both a mixed strategy and a signaling scheme. The role of the signaling scheme is to reveal different information to different followers about the realization of the leader's strategy, and in doing so to improve the leader's utility.

We restricted attention to a state of nature drawn from a common prior, and a principal who simply discloses information to agents without soliciting information. Two recent works have explored relaxing these assumptions for the Bayesian persuasion model described in Section 2. [Alonso and Camara 2014] characterize the optimal signaling scheme when the sender and receiver have different prior distributions. [Kolotilin et al. 2015] consider a privately-informed receiver, and a sender who first solicits the agent's information and then selectively signals his own information. Under some assumptions, they characterize the optimal combined policy. To our knowledge, neither of these two generalizations of the Bayesian persuasion model has been explored algorithmically.

Finally, we mention that a number of related, but importantly different, models have a long history in the economics community, yet to our knowledge have not been explored from a computational perspective. Most notably, the literature on *cheap talk* does away with the power of commitment, and much of that work analyzes the equilibria of cheap talk games. Also, the literature on *verifiable information* restricts the principal to signals which are meaningful and "honest" in a precise sense. We refer the reader to the survey by [Sobel 2010] which compares and contrasts a variety of these models.

6. CONCLUSIONS

The past few years have seen an explosion of interest in understanding the effects of information on strategic interactions. Since information structure design, like traditional mechanism design, is fundamentally algorithmic, it was inevitable that computer science would have much to say on the topic. This survey illustrates how asking computational questions has led to new structural insights in this area. Moreover, we believe that information structure design has grown into a deep theory waiting for an application, and simple and efficient algorithms will pave the way for such impact.

In addition to the open problems mentioned in Sections 2 through 4, there is much work to be done beyond the three basic models. Specifically, Section 5 suggests a number of related or more expressive models which have not been subjected to rigorous examination through a computational lens. We also believe there is room for experimental work to test the predictive power of this theory. Specifically, to what extent is signaling effective when deployed against human agents? [Azaria et al. 2015] examine this question in a single-agent setting (a slight variant of the Bayesian persuasion model from Section 2), and their findings are encouraging in the pair of domains they consider. More experimental validation of the predictions of information structure design, both in single and multiple agent settings, remains to be done.

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