Settling the Complexity of Computing Approximate Two-Player Nash Equilibria

AVIAD RUBINSTEIN
UC Berkeley

In our recent paper [Rubinstein 2016] we rule out a PTAS for the 2-Player Nash Equilibrium Problem. More precisely, we prove that there exists a constant $\epsilon > 0$ such that, assuming the Exponential Time Hypothesis for PPAD, computing an ϵ-approximate Nash equilibrium in a two-player $n \times n$ game requires time $n^{\log^{1-o(1)} n}$. This matches (up to the $o(1)$ term) the algorithm of Lipton, Markakis, and Mehta [Lipton et al. 2003].

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Introduction

Equilibrium is a ubiquitous assumption for modeling non-cooperative game, as well as markets, traffic, biospheres, and many other systems. Once players are at equilibrium they have no incentive to deviate. But how do they arrive at an equilibrium in the first place? This question has been studied by economists for over half a century (e.g. [Brown 1951; Robinson 1951; Lemke and Howson 1964; Scarf 1967]), but a general recipe is yet to be found. In recent decades, it was considered under the *Lens of Computation* by looking at the surrogate: "is there an efficient algorithm for computing an equilibrium?" Breakthrough results of [Daskalakis et al. 2009] and [Chen et al. 2009] proved that such an algorithm does not exist (assuming PPAD $\neq$ P). Furthermore, if a central, specially designed algorithm fails to find an equilibrium, it is even less likely that distributed, selfish agents will naturally converge to one. This puts the entire solution concept in doubt.

For the past decade, the main open question in this field was whether the computational intractability results extend to approximate equilibria. We had good reasons to hope that they don't, i.e. that two-player Nash admits a PTAS: there was a series of improved approximation factors in polynomial time [Kontogiannis et al. 2009; Daskalakis et al. 2009; 2007; Bosse et al. 2010; Tsaknakis and Spirakis 2008] and several approximation schemes for special cases [Kannan and Theobald 2007; Daskalakis and Papadimitriou 2009; Alon et al. 2013; Barman 2015]. Yet most interesting are two inefficient algorithms for two-player Nash:

—the classic Lemke-Howson algorithm [Lemke and Howson 1964] finds an exact Nash equilibrium in exponential time; and

Authors' addresses: aviad@eecs.berkeley.edu

—a simple algorithm due to [Lipton et al. 2003] finds an ϵ-Approximate Nash Equilibrium in time $n^{O(\log n)}$.

Although the Lemke-Howson algorithm takes exponential time, it has a special structure which places the problem inside the complexity class PPAD [Papadimitriou 1994]; i.e. it has a polynomial time reduction to the canonical problem EndOfALine:

Definition 1 *(*EndOfALine *[Daskalakis et al. 2009]).* Given two circuits S and P, with m input bits and m output bits each, such that $P(0^m) = 0^m \neq S(0^m)$, find an input $x \in \{0,1\}^m$ such that $P(S(x)) \neq x$ or $S(P(x)) \neq x \neq 0^m$.

Proving hardness for problems in PPAD is notoriously challenging because they are *total*, i.e. they always have a solution, so the standard techniques from NP-hardness do not apply. By now, however, we know that exponential and polynomial approximations for two-player Nash are PPAD-complete [Daskalakis et al. 2009; Chen et al. 2009], and so is ϵ-approximation for games with n players [Rubinstein 2015b].

ϵ-approximation for two-player Nash is unlikely to have the same fate: otherwise, the quasi-polynomial algorithm of [Lipton et al. 2003] would refute the Exponential Time Hypothesis for PPAD:

Hypothesis 2 (ETH for PPAD [Babichenko et al. 2016]). *Solving* EndOfALine *requires time* $2^{\tilde{\Omega}(n)}$.[1]

Thus the strongest hardness result we can hope to prove (given our current understanding of complexity) is a quasi-polynomial hardness that sits inside PPAD, and this is precisely the main result of [Rubinstein 2016]:

Theorem 3 (2 players [Rubinstein 2016]). *There exists a constant $\epsilon > 0$ such that, assuming ETH for* PPAD*, finding an ϵ-Approximate Nash Equilibrium in a two-player $n \times n$ game requires time $T(n) = n^{\log^{1-o(1)} n}$.*

Quasi-fine-grained Complexity

Two-player Nash equilibrium belongs to a growing class of fundamental problems that admit a quasi-polynomial time approximation algorithms, and also have matching conditional lower bounds on the running time. Those include problems of relevance to the SIGecom community, such as ϵ-best ϵ-Nash equilibrium [Braverman et al. 2015; Deligkas et al. 2016], Densest k-subgraph [Braverman et al. 2017; Manurangsi 2016], signaling in a zero-sum game [Rubinstein 2015a; Bhaskar et al. 2016], and community detection [Rubinstein 2017].

For all of those problems, the birthday repetition framework [Aaronson et al. 2014] gives a reduction size of $N \approx 2^{\sqrt{n}}$. Assuming the exponential time hypothesis (ETH) [Impagliazzo et al. 2001], approximating 3-SAT requires time $T(n) \approx 2^n \approx N^{\log N}$; hence the quasi-polynomial lower bound. The same blowup in instance size occurs in the proof of Theorem 3.

Unfortunately, it is not clear how to apply the birthday repetition framework to Nash equilibrium because we don't have an equivalent of the PCP Theorem for

[1] As usual, n is the size of the description of the instance, i.e. the size of the circuits S and P.

PPAD. (But recently [Babichenko et al. 2016] conjectured what a "PCP for PPAD" could look like - and proving it remains an important open problem.) The actual proof of Theorem 3 in [Rubinstein 2016] circumvents this obstacle and does not explicitly use birthday repetition. As a result, it is quite involved and requires tools from the studies of PCP, locally decodable codes, pseudorandomness, etc. In particular, this is the first time that such ideas are used for hardness of approximation inside PPAD.

Can almost everyone be almost happy?

An ϵ-Approximate Nash Equilibrium (ϵ-ANE) is a (mixed) strategy profile for which each player plays an ϵ-best response; i.e. she can gain at most (an additive) ϵ by deviating. This is the standard and most-studied notion of approximate Nash equilibrium, and it is indeed very natural for two-player games. However, for some settings with *many players*, requiring that the ϵ-best response condition holds for *every* player may be too restrictive. Recently, [Babichenko et al. 2016] introduced a more relaxed notion of (ϵ, δ)-WeakNash Equilibrium, where we only require that a $(1 - \delta)$-fraction of the players play ϵ-best response, while the remaining δ-fraction may play arbitrarily.

En route to proving Theorem 3 we obtain impossibility results for the latter WeakNash relaxation:

THEOREM 4 (n PLAYERS [RUBINSTEIN 2016]). *There exist constants $\epsilon, \delta > 0$ such that finding an (ϵ, δ)-WeakNash Equilibrium...*

Query Complexity. *requires $2^{\Omega(n)}$ oracle calls to the payoff tensor; and*

Computational Complexity. *is PPAD-hard given a succinct description of the payoff tensor.*

We note that the former result on query complexity resolves an open question posed by Hart and Nisan [Hart and Nisan 2013], Babichenko [Babichenko 2016], and Chen et al. [Chen et al. 2017]. Furthermore, in subsequent joint work with Babichenko [Babichenko and Rubinstein 2016], we extend this result to a lower bound on **communication complexity** (where each player knows her own utilities).

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