Efficiency and Price Discovery in Multi-item Auctions

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Distributed multi-item auctions offer great opportunities for integrating fragmented online auction markets into larger markets with more efficient outcomes. We extend the theory of multi-item ascending auctions in a multi-unit demand scenario. We show that a simple greedy bidding strategy results in efficient allocation and unique prices. We also show that the strategy constitutes a Nash Equilibrium of the system with single unit demand. We discuss the implications of our results for the design of auctions on the Internet.

Additional Key Words and Phrases: Auctions, Multi-item Auctions, Allocation, Efficiency, Bidding Strategy, Local Greedy Bidding, LGB, Nash Equilibrium, Dominant Strategy.

1. INTRODUCTION

Recent advances in Internet and electronic commerce technologies have made online auctions very popular. The present day auction markets on the Internet are however fragmented into a large number of very small markets selling similar items, potentially leading to inefficient outcomes. In general, it is desirable to take into account the presence of all items being sold simultaneously and the presence of all competing bidders in determining the allocation and prices. We define a system of distributed multi-item online auctions and show that if items are "substitutes" for bidders, then a simple greedy bidding strategy results in efficient allocation. Thus, we provide a means for integrating fragmented auction markets on the Internet without requiring a centralized clearing house.

We consider a system of ascending (open outcry) auctions where each auction is conducted independently, except that all auctions open and close at the same time. Each auction is to sell one or more copies of an item. Without loss of generality, we can equivalently consider a system with m items, each on a separate auction,

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with a $m \times 1$ vector of reserve prices P^r. The reserve price for an item j equals the valuation of the auctioneer of j for the item j. We consider n bidders in the system. Each bidder i wants up-to k_i items. For each item j, a bidder i has an independent private valuation v_{ij}. If a bidder does not want an item j, then $v_{ij} = 0$. The system of Simultaneous Independent Ascending auctions can thus be represented as $SIA(V, P^r)$ where V is the $n \times m$ matrix of bidders' valuations. To ensure that the auctions terminate in a finite number of steps, a new bid must exceed an existing bid by ϵ. If bidders are software agents, ϵ can be set to a very small value. The state of the system at any time is given by the vector of current prices (winning bids) for the m items, P and their corresponding winners. For each auction, the bidder with the highest bid at the close of the auction is declared the winner and pays an amount equal to its bid to get the item.

The underlying market theory dates back to [Shapley and Shubik 1972] who considered an assignment market where each participant wants one item and places a monetary value on each item in the market. It was shown that if the valuations of all participants are known then the model always has an equilibrium with a unique smallest price vector P. [Bikhchandani and Mamer 1997] extended the results of Shapley and Shubik to an economy where a participant may want a subset of items instead of a single item. The participants' preferences are specified by non-decreasing monetary values for each subset of items, i.e., if $T \subset S$ then $value(T) \leq value(S)$. They derived the necessary and sufficient conditions for the existence of market clearing prices and showed that if market clearing prices existed, the corresponding allocation would be efficient.

Do there exist auction mechanisms, which can efficiently allocate multiple items when the conditions specified in [Bikhchandani and Mamer 1997] are met? In particular, do there exist mechanisms that can be implemented on the Internet to create large integrated global markets?

[Leonard 1983] and [Demange and Gale 1985] showed the existence of a truth-telling dominant strategy for sealed bid multi-item auction of substitutes with single unit demand which achieves the equilibrium suggested in [Shapley and Shubik 1972]. However, such an auction mechanism does not lend itself to decentralized auctions on the Internet.

[Milgrom 1998] considered a multi-round sealed bid auction for multiple items and showed that if the bidders bid 'straightforwardly' (place bids in the next round for the subset of items that maximizes their current surplus), then a competitive equilibrium results which is efficient. However, the straightforward bidding strategy is restrictive since it requires that all bidders be present right from the beginning of the auction and place a bid in each round. In practical Internet auctions, it should be possible for bidders to join any time and stay inactive for some parts of the auction.

[Demange et al. 1986] considered an ascending auction where the bidders want only a single item out of a set of heterogeneous items. They showed that *if* the bidders announced honestly at each stage the item whose value to the bidder exceeds its current price by the maximum amount, the auction mechanism nearly converges to the smallest equilibrium price vector P with each component of P being within $l\epsilon$ (where l is the lesser of the number of items and bidders and ϵ is the minimum required bid increment) of the corresponding smallest equilibrium price. [Miyake

1998] showed that under assumptions of monotonicity and continuity on the bidders' preferences, the honest strategy assumed by [Demange et al. 1986] converges to a dominant strategy in the limit of the bid increment approaching zero. [Wellman et al.] further showed that the inefficiency in the resulting allocation is bounded by at most $l(l+1)\epsilon$.

Multi-item auctions are not well studied in the case of multi-unit demand. The existing work on multi-item ascending auctions for single unit demand seems to suggest that the average inefficiency per item (in a system of l items) is proportional to l. Such a system does not scale well for large integrated auction markets on the Internet. We consider multi-item auctions with multi-unit demand and derive a tighter bound on inefficiency which alleviates this problem. Extending the work of [Demange et al. 1986], we obtain some corresponding results on prices for the case of multi-unit demand.

We also look at the incentive properties of multi-item ascending auctions with a single unit demand in the presence of deviant bidders who do not adhere to the assumptions used by [Miyake 1998]. We show that under these conditions, the honest bidding strategy assumed by [Demange et al. 1986] ceases to be a dominant strategy. However, we show that it still constitutes a Nash Equilibrium for the system. Finally, we discuss the implications of our results for Internet auction markets.

2. SYSTEM PROPERTIES WITH GREEDY BIDDING

Define the surplus of a bidder on an item to be the amount by which its valuation for the item exceeds the current price. Consider a generalization of the bidding strategy suggested by [Demange et al. 1986] for multi-unit demand. A bidder requiring up-to k items orders the items with positive surplus in the decreasing order of their surplus and places bids on the first k items. On being outbid, the bidder again selects the k highest (and positive) surplus items and places bids on all items from amongst those where it is not winning. We denote this strategy by Local Greedy Bidding (LGB) strategy. Since the bid increments and bidders' valuations are finite, the bidding would terminate in a finite number of steps.

Given an allocation of items to bidders, the total surplus created in the system is defined by: $S = \sum_{j=1}^{m}(v_{w(j)j} - p_j^r)$ where $w(j)$ is the winner of item j. This surplus is independent of the prices and depends only on the assignment of the items to bidders. If the bidders follow LGB strategy, the resulting allocation is nearly efficient, i.e., maximizes the total surplus for the system. Further, the inefficiency for the system is bounded by $l\epsilon$ where l is the number of items sold. This is formally stated in the following theorem.

THEOREM 1 (EFFICIENT ALLOCATION). *If S_{OPT} is the maximum total surplus in the system $SIA(V, P^r)$ under any allocation, and S_{LGB} is the total surplus when the bidders follow the LGB strategy then, $S_{OPT} - S_{LGB} \leq l\epsilon$.*

The proof for single unit demand is provided in [Bansal and Garg 2000]. The proof for the case of multi-unit demand would be included in its subsequent revision. This theorem implies that the average inefficiency per item for a system having sold l items on simultaneous auctions is a small constant, independent of the number of bidders and items in the system. Therefore, the system can scale to arbitrarily large integrated auction markets on the Internet without losing on efficiency in the process.

Since the reserve prices are fixed, the allocation is such that the sum of valuations of the winners for their respective items is close to the maximum such sum possible for any allocation feasible for the system. This can be regarded as a generalization of the simple open outcry auction for multiple identical copies of a single item, where the copies are allocated to the bidders with the highest valuations.

While the prices of individual items do not affect the total surplus for the system, they determine the distribution of the system surplus amongst bidders and auctioneers, thus influencing their behaviour and participation in the system. In general, competitive price discovery is considered as important in a market mechanism as efficiency.

For the single unit demand, [Demange et al. 1986] showed that the greedy bidding strategy results in prices that approximate the unique minimum competitive price vector within $2l\epsilon$. We obtain similar results for the case of multi-unit demand.

THEOREM 2 (COMPETITIVE PRICES). *Consider the system $SIA(V, P^r)$ with the bidders following the LGB strategy. Let P^1 and P^2 be the final price vectors for two different outcomes of the system. Then for all items j, $| p_j^1 - p_j^2 | \leq cl\epsilon$ where c is a constant.*

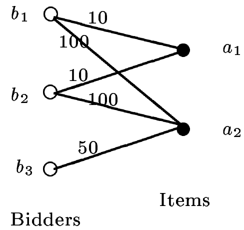
An alternate proof for single unit demand with a bound of $3l\epsilon$ is also provided in [Bansal and Garg 2000]. The proof for the case of multi-unit demand would be included in the next revision.

A useful property of the given system with LGB strategy is that for any subset of the sold items, all the prices are determined by the bidders whose demand is not completely satisfied from within the set. For the specific case of single unit demand, this implies that all the prices for a set are determined by bidders who do not win any items from the set (but have highest losing bids on some items in the set). These characteristics resemble those observed in $M + 1^{st}$ price auctions where the highest non-winning bid determines the price.

When bidders follow LGB strategy and the bid increment is sufficiently small we can show that as the number of bidders in the system (or the units demanded by them) increases, the new prices are guaranteed to rise or remain the same. Therefore, the auctioneers would like to expose their items to as many bidders as possible. On the other hand, as more items are introduced into the system, the new prices would fall or remain the same. This provides an incentive for the bidders to want more auctions to be included into the system.

It is worthwhile to examine the characteristics of the strategic behaviour of bidders in the context of the present system. In general, in the presence of multi-unit demand, a surplus maximizing bidder may withhold a part of its demand to obtain lower prices and better surplus. Therefore, in the present context, revealing true demand by way of LGB may not be a dominant strategy for bidders who want more than one item. However, for the case of single unit demand, [Miyake 1998] has shown that if the set of strategies that the bidders can follow is suitably restricted, LGB turns out to be a dominant strategy.

It is not clear whether LGB continues to be a dominant strategy or even constitute a Nash Equilibrium in the presence of deviant bidders who do not adhere to the as-



Bidder b_1's strategy:
if $(0 < p_1 < 10)$ then
bid on item a_1
else if $((p_1 = 0)$ and $(p_2 < 100))$ then
bid on item a_2
else do not bid.

Bidder b_2's strategy:
place the first bid on item a_1
if (outbid and $p_2 < 100$) then
bid on item a_2
else do not bid.

Bidder b_3's strategy:
if $(p_2 < 50)$ then
bid on item a_2
else do not bid.

Fig. 1. An Example where LGB is not a Dominant Strategy

sumptions used by Miyake. It turns out that in a general setup where no restrictions are placed on the strategies that the bidders can follow, LGB is not a dominant strategy. To see this, consider a system $SIA(V, P^r)$ as shown in Figure 1 with $V(b_1) = (10, 100)$, $V(b_2) = (10, 100)$, $V(b_3) = (0, 50)$ and $P^r = (0, 0)$. Assume that $\epsilon = 1$. Bidder b_1 follows the strategy: `if` $(0 < p_1 < 10)$ `then bid on item` a_1 `else if` $((p_1 = 0)$ `and` $(p_2 < 100))$ `then bid on item` a_2 `else do not bid.` Bidder b_2 follows the strategy: `place its first bid on item` a_1`. On being outbid, if` $(p_2 < 100)$ `then place a bid on item` a_2. Bidder b_3 uses the strategy: `if` $(p_2 < 50)$ `then place bid on item` a_2.

With these set of strategies, bidder b_2 will either get item a_1 at a price of 1 (with a surplus of 9) or item a_2 at a price of 51 (with a surplus of 49). However, if bidder b_2 were to follow LGB strategy, it would get item a_2 at a price of either 90 or 91 (with surpluses of 10 and 9 respectively) or item a_1 at a price of 1 (with a surplus of 9). Bidder b_2 may realize significantly greater surplus by following a strategy other than LGB when other bidders follow arbitrary strategies as indicated.

Therefore, LGB is not a dominant strategy for the system under general conditions. However, it does constitute a Nash Equilibrium for the case of single unit demand as indicated below.

Theorem 3 (Nash Equilibrium). *Consider any outcome of $SIA(V, P^r)$ with bidders having single unit demand and following LGB strategy. Let s_i be the surplus of bidder i. Bidder i cannot obtain a surplus greater than $s_i + 3l\epsilon$ by unilaterally following some other strategy. Therefore the system approaches Nash Equilibrium in the limit when the bid increment approaches zero.*

The proof is provided in [Bansal and Garg 2000]. For multi-unit demand, the LGB strategy does not constitute a Nash Equilibrium due to the incentive to withhold demand.

For single unit demand, the bound of $2l\epsilon$ on variation in prices is tight. It is possible to construct an instance of SIA and two bidding sequences leading to different outcomes such that the price of an item in these two outcomes differs by $2l\epsilon$. This raises some concerns in a large scale implementation because it seems to suggest that the maximum variation in the price of an item may be of the order of the size of the system. In a large system, if $2l\epsilon$ becomes significant, then bidders may adopt other strategies to maximize their surplus.

For single unit demand, it can be shown (see [Bansal and Garg 2000]) that an instance of SIA system can be decomposed into smaller independent SIA subsystems, where the prices within a subsystem can be determined independent of the other subsystems. The maximum variation in the price of an item can be bounded by the size of the subsystem in which the item is present. Therefore, even in a very large system the price variations in different outcomes may turn out to be very small if the decomposed subsystems are small in size.

3. CONCLUSIONS

The present work extends the theory of multi-item ascending auctions to address some issues that become important in a large scale online implementation. The multi-item ascending auctions can be implemented in the form of a decentralized mechanism consisting of simultaneous but independent ascending (open outcry) auctions on the Internet. They can therefore serve to create large integrated auction markets on the Internet. We show that the use of a simple truth-telling greedy bidding strategy (LGB) leads to equilibrium with many desirable properties.

The existing Internet auctions attempt to maximize the surplus in each auction (or a small auction market) separately and in the process leads to a system outcome which may be inefficient. We saw that use of the LGB strategy in SIA leads to nearly efficient allocation with the average inefficiency per item bounded by the size of the bid increment. The additional surplus in SIA with LGB bidding comes from the ability to take into account the valuations of all bidders for all items in deciding the allocation.

LGB strategy (with single unit demand) constitutes a Nash equilibrium for the system SIA under very general conditions. Therefore, given the items and bidders, no auctioneer can increase its surplus by selling its item at a higher price and no single bidder can get a higher surplus by following any other strategy. The resulting allocation of surplus between auctioneers and bidders is competitive and hence fair. Under somewhat restrictive conditions, LGB is also a dominant strategy.

For the bidders, the ability to bid across multiple simultaneous auctions results in greater choice. For the auctioneers, simultaneous auctions with bidders bidding across them implies more competition for their items. For the system as a whole, it implies participation by all bidders in auctions of all items, thereby yielding the surplus maximizing allocation of items to bidders.

A very desirable property of SIA is that each auctioneer can conduct its own auction independently as a simple open outcry auction. The only requirement is that the auctions be simultaneous. To achieve this, various auction sites only need to group auctions of similar items and agree on common start and end times. Another desirable property is that LGB is a fairly simple strategy and needs only the valuations of a bidder and the current price vector of auctions for bidding. Therefore, very simple software agents can implement this strategy and execute it for bidding across multiple auctions on an auction site and also across multiple auction sites. Thus, the mechanism provides for the formation of large integrated auction markets on the Internet without requiring the auctioneers and bidders to cooperate to any significant extent.

SIA with LGB strategy may be seen as a generalization of multi-unit ascending auctions to independent ascending auctions of heterogeneous items operated in a decentralized fashion.

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