Bayes-Nash Equilibria For m^{th} Price Auctions With Multiple Closing Times

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In this paper we compute Bayes-Nash equilibria for first price single unit auctions and m^{th} price multi unit auctions, when the auction has a set of possible closing times, one of which is chosen randomly for the auction to end at. Thus the auctions have one or more rounds of sealed bids. We compute such equilibria for a wide range of assumptions and demonstrate the method used by an agent to generate these strategies.

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1. INTRODUCTION

Auctions are becoming an increasingly popular method for transacting business either between individuals over the Internet (e.g. eBay) or even between businesses and their suppliers. Auction theory provides us with some simple equilibria mostly for the case that a single item is being sold or bought (see [Krishna 2002]). In order to examine the strategic choices in a vastly more complex game, the Trading Agent Competition was introduced (see [Wellman et al. 2001]). Different agents used different approaches to the problem (for some of them see [Stone et al. 2002], or [Greenwald and Boyan 2001]). In [Vetsikas and Selman 2003] the authors presented a principled methodology for systematically exploring the space of bidding strategies for a complex game like TAC, where it is not possible to find an equilibrium solution. To handle the complexity the authors decompose the problem into sub-problems[3]; then the various strategies (for each sub-problem) are recombined to generate the strategy that the agent uses.

In this paper, we concentrate on one of those sub-problems, the purchase of hotel rooms. These auctions have a set of possible closing times, one of which is chosen randomly for the auction to end at. They can be decomposed into one or more rounds, each of which is defined by the intervals between possible closing times, and during which agents submit sealed bids. In this work, we present the basic steps towards computing the Bayes-Nash equilibria that exist in such cases and compute several novel equilibria for these auctions.

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[3]Each sub-problem focuses only on one auction (either for plane tickets, hotel rooms, or entertainment tickets).

In particular, we present the analysis for the case when the auction has $R = 2$ rounds. In section 3, we compute equilibria for the single unit (first price) auction, whereas, in section 4, we compute equilibria for the multi-unit (m^{th} price) auction. In each case, we initially examine the strategic interactions that exist in each round separately, and then use this knowledge in order to analyze the entire multi-round auction. As our stated goal is to make use of the discovered equilibria in order to generate strategies that our agents can use when participating in TAC, in [Vetsikas et al. 2007] we extend this work to fully analyze the multi-round ($R \geq 2$) case. This extension makes substantial usage of the basic ideas and theorems presented in this paper. For example, the proof of theorem 4.1, which examines the strategic interactions of one round of a multi-unit auctions, and which is only presented in this paper, is used in [Vetsikas et al. 2007] in order to generate the equilibria for the multi-unit auction with any number of rounds.

2. PROBLEM SETUP

The part of the TAC game that we are interested in are the hotel room auctions. There are $m = 16$ rooms available each night at each of the two available hotels. Rooms for each of the days are sold by each hotel in separate, ascending, multi-unit, 16^{th}-price auctions. These auctions close at randomly determined times and more specifically a random auction will close every minute throughout the game. No prior knowledge of the closing order exists and agents may not resell rooms. Between closing times the agents may place bids, but these are not opened until the next possible closing time; hence each round that takes place between consecutive closing times is a sealed bid auction.

We assume that N risk-neutral agents wish to buy 1 unit of a certain good each. An independent seller sells m units of the desired good in an m^{th} price auction, i.e. the good is sold to the agents which submitted the m highest bids at a price equal to the lowest winning bid. The agents have valuations (utilities) u_i which are i.i.d. with probability distribution $F(u)$ in the first round. Each agent know its own valuation and the distribution $F(u)$. There can be a second round with probability $(1-p)$, where p is known. If a second round does exist, the agents have new i.i.d. utilities $\widetilde{u}_i$ drawn from some distribution $H(u)$ and can submit new bids as long as they are greater or equal to the bid price from the end of the first round. The assumptions about what each agent i knows about its utility $\widetilde{u}_i$ at the start of the game determine the different cases that we examine.

—$\widetilde{u}_i$ can be assumed to be known (and in fact in some of the theorems that follow it is presumed that $\widetilde{u}_i \simeq u_i$, i.e. that $\widetilde{u}_i$ and u_i have similar values); this is reasonable in the case of TAC because usually there is a correlation between the valuation of the same room over the course of the game.

—The agent might not know anything about $\widetilde{u}_i$ other than that it is drawn from $H(u)$ (this is the same information that it has about the other agent's valuations).

—The agent might know something about $\widetilde{u}_i$. In TAC knowing the utilities at the previous rounds can allow the agent to compute that $\widetilde{u}_i$ is drawn from a more "tight" and accurate distribution $G(u)$ instead of $H(u)$. One example of this is that the utility at a later round is highly unlikely to decrease, so values $\widetilde{u}_i < u_i$ can be discarded.

The last rule of the auction is that agents may not subtract bids; this means that, if its utility drops in a later round below the current bid price (which we will denote Q), the agent cannot withdraw its previous bid, but it can adjust it to the current bid price. The effect of this is that if the price Q is high enough that fewer than m agents have utilities $\widetilde{u}_i \geq Q$, the rest of the rooms are sold to a random selection of the winners of the previous round which have $\widetilde{u}_i < Q$ and these agents lose money.

We will use the following functions in the theorems:

$\Phi(x) = \sum_{i=0}^{m-1} C(N-1,i) \cdot \big(F(x)\big)^{N-1-i} \cdot \big(1-F(x)\big)^i$

$Y(x) = \sum_{i=0}^{m-2} C(N-1,i) \cdot \big(F(x)\big)^{N-1-i} \cdot \big(1-F(x)\big)^i$

$\widetilde{\Phi}(x) = \sum_{i=0}^{m-1} C(N-1,i) \cdot \big(H(x)\big)^{N-1-i} \cdot \big(1-H(x)\big)^i$

$\widetilde{Y}(x) = \sum_{i=0}^{m-2} C(N-1,i) \cdot \big(H(x)\big)^{N-1-i} \cdot \big(1-H(x)\big)^i$

In the case that $m=1$, it is $\Phi(x) = \big(F(x)\big)^{N-1}$, $\widetilde{\Phi}(x) = \big(H(x)\big)^{N-1}$ and $Y(x) = \widetilde{Y}(x) = 0$.

Before we proceed with our analysis we need the following information, found in numerical analysis textbooks (see [Atkinson and Han 2004]).

THEOREM 2.1. *Let $f(x,z)$ and $\frac{\partial f(x,z)}{\partial z}$ be continuous functions of x and z at all points (x,z) in some neighborhood of the initial point (x_0, Y_0). Then there is a unique function $Y(x)$ defined on some interval $[x_0-\alpha, x_0+\alpha]$, satisfying*

$Y'(x) = f(x, Y(x)), \quad \forall x : x_0 - \alpha \leq x \leq x_0 + \alpha$ *and*

$Y(x_0) = Y_0$

All the equilibria when $p \neq 1$ are the solutions of differential equations of the form described by theorem 2.1. This theorem guarantees the existence and unique solvability of the initial value problem for those differential equations, which in turn means that the equilibrium does exist and is unique. We may not know a closed form solution, but a numerical solution can be easily calculated. The method that we decided to use is a 4^{th} order Runge-Kutta method with variable step size; this is one of the most commonly used methods. The requirement is that the function $f(x,z)$ and several (this depends on the order of the Runge-Kutta method) of its derivatives be continuous in the interval for which the solution is computed.

3. BAYES-NASH EQUILIBRIA FOR A SINGLE UNIT AUCTION

In this section, we present the equilibium analysis for the case when a single ($m = 1$) item is sold to the agent which submitted the highest bid at a price equal to his bid. For theorems 3.1 and 3.2, we assume that in the second round the utilities are drawn from $F(u)$ and that $\widetilde{u}_i \simeq u_i$. Each agent i submits a bid v_i in the first round. It is $Q = 0$, if no bids were placed, which is the case at the beginning of the first round, whereas $Q > 0$ equals the current bid price in the beginning of the second round. We compute a Bayes-Nash equilibrium $g(u)$ that maps utilities u_i to bids v_i.

In the case of $p = 1$ (only one round) and $Q = 0$, we know from auction theory (e.g. [Krishna 2002]) that each risk-neutral agent i with valuation u_i should bid

$$g(u_i) = u_i - \frac{1}{\big(F(u_i)\big)^{N-1}} \cdot \int_0^{u_i} \big(F(\omega)\big)^{N-1} \cdot d\omega \tag{1}$$

THEOREM 3.1. *If the starting price is $Q \geq 0$ and the bidding lasts for exactly one round ($p = 1$) the equilibrium strategy is*

$$g(u_i) = u_i - \frac{\int_Q^{u_i} \big(F(\omega)\big)^{N-1} \cdot d\omega}{\big(F(u_i)\big)^{N-1}} \tag{2}$$

PROOF. The proofs for this theorem, as well as most of the missing proofs from this paper are presented in [Vetsikas et al. 2007]. Unabridged versions of all the proofs are presented in [Vetsikas 2005].

THEOREM 3.2. *If the starting price is $Q = 0$, a second round of bidding exists with probability $(1-p)$ ($p \neq 0, 1$) and the utility of the agents in the second round is drawn from the same distribution $F(u)$ (and each agent i in fact has utility of a similar value to the utility u_i of the first round) then the equilibrium strategy is the solution of the differential equation*

$$\big(u_i - g(u_i)\big) \cdot \frac{\Phi'(u_i)}{g'(u_i)} = \Phi(u_i) \cdot \Psi(g(u_i)) \tag{3}$$

where $\Psi(x) = 1 + \frac{1-p}{p} \cdot \big(F(x)\big)^{N-1}$,
and the boundary condition is $g(0) = 0$.

As a special case we can examine this equation when only $N = 2$ agents participate and their valuations $u_i \sim U[0,1]$, which means that $F(u) = u, \forall u \in [0,1]$. We need to compute $v_i = g_p(u_i), \forall u_i \in [0,1]$. Equation 3 becomes:

$$g_p^{'}(u_i) = \frac{u_i - g_p(u_i)}{u_i \cdot \big(1 + \frac{1-p}{p} \cdot g_p(u_i)\big)} \tag{4}$$

Even this equation, which is the simplest form that we can have for the two round auction, has no known closed form solution.
However, we can at least remove the parameter p from this computation. We can easily verify that $g_p(u) = \frac{p}{1-p} \cdot \widetilde{g}(\frac{1-p}{p} \cdot u)$, where $\widetilde{g}(u)$ is the solution of d.e. $\widetilde{g}^{'}(u) = \frac{u - \widetilde{g}(u)}{u \cdot (1 + \widetilde{g}(u))}$.

4. BAYES-NASH EQUILIBRIA FOR A MULTI-UNIT AUCTION

In this section, we present the equilibrium strategies for multi-unit auctions ($m > 1$). The goods are sold to the agents which submitted the m highest bids at a price equal to the lowest winning bid. For theorems 4.1 and 4.2, we assume that in the second round the utilities are drawn from $F(u)$ and that $\widetilde{u}_i \simeq u_i$, i.e. that the utility in the next round is similar to the one in the current round. In theorem 4.3, we assume that the agent knows that its own utility $\widetilde{u}_i$ is drawn from $G(u)$ and everyone else's from $H(u)$.

THEOREM 4.1. *If the starting price is $Q \geq 0$ and the bidding lasts for exactly one round ($p = 1$) the equilibrium strategy is*

$$g(u) = u - \frac{e^{\int_Q^u \frac{-Y'(\omega)}{\Phi(\omega) - Y(\omega)} \cdot d\omega}}{\Phi(u) - Y(u)} \cdot \int_Q^u \frac{\Phi(z) - Y(z)}{e^{\int_Q^z \frac{-Y'(\omega)}{\Phi(\omega) - Y(\omega)} \cdot d\omega}} \cdot dz \tag{5}$$

PROOF. $Q \geq 0$ because some bids may have already been placed, and thus (i) some agents might have stopped participating in the auction, since the current price Q exceeds their private valuation u_i, and (ii) the probability distribution of the valuations $F(u)$ has changed, since now we know that the valuation of agents that still participate is $u_i \geq Q$. The new probability distribution is
$F_Q(u) = Prob[U \leq u | U \geq Q] = \frac{Prob[U \leq u \wedge U \geq Q]}{Prob[U \geq Q]} = \frac{F(u) - F(Q)}{1 - F(Q)}$. Therefore

$$F_Q(u) = \frac{F(u) - F(Q)}{1 - F(Q)},\ if\ u \geq Q\ \&\ F_Q(u) = 0,\ if\ u < Q \tag{6}$$

We also know the probability π_k of the event that exactly $k \in [1, N]$ agents participate in the auction at price Q; it is the probability that exactly $k-1$ of the other agents' valuations[4] u_i are $u_i \geq Q$, which is (see lemma 6.2).

$$\pi_k = C(N-1, k-1) \cdot \big(F(Q)\big)^{N-k} \cdot \big(1 - F(Q)\big)^{k-1} \tag{7}$$

The probability distribution of an agent's bid is: $Prob[V \leq v] = Prob[g(U) \leq v] = Prob[U \leq g^{-1}(v)] = F(g^{-1}(v))$.

Therefore the probability distribution of the $(m-1)^{th}$ and m^{th} highest bids ($m > 1$), named $B^{(m-1)}$ and $B^{(m)}$ respectively, among all other $N-1$ agents are:

$Prob[B^{(m-1)} \leqslant v] = \widehat{Y}(g^{-1}(v))$ and $Prob[B^{(m)} \leqslant v] = \widehat{\Phi}(g^{-1}(v))$,

where $\widehat{\Phi}(x) = \sum_{i=0}^{m-1} C(k-1, i) \cdot \big(F_Q(x)\big)^{k-1-i} \cdot \big(1 - F_Q(x)\big)^{i}$ and

$\widehat{Y}(x) = \sum_{i=0}^{m-2} C(k-1, i) \cdot \big(F_Q(x)\big)^{k-1-i} \cdot \big(1 - F_Q(x)\big)^{i}$ (see lemma 6.1).

If $k \leq m$ then the expected utility is: $EU_i(v_i|\#agents = k) = u_i - Q$.

If that is not the case, meaning that $k \geq m$ and $B^{(m-1)} \geq Q$, then:

If $B^{(m)} > v_i$, then the agent gets utility 0 (does not win). If $B^{(m-1)} > v_i \geq B^{(m)}$, then the agent submitted the m^{th} price, so it gets 1 unit (of the m available units) and pays v_i getting a utility of $u_i - v_i$. If $v_i \geq B^{(m-1)} \geq Q$, then the agent gets 1 unit and pays the m^{th} price, which is $B^{(m-1)}$, getting a utility of $u_i - B^{(m-1)}$. The expected utility is

$$\begin{aligned}
&EU_i(v_i|\#agents = k) = (u_i - v_i) \cdot Prob[B^{(m-1)} > v_i \geq B^{(m)}] \\
&\qquad + \textstyle\int_Q^{v_i} (u_i - \omega) \cdot Prob[B^{(m-1)} = \omega] \cdot d\omega = \\
&(u_i - v_i) \cdot \big(Prob[B^{(m)} \leq v_i] - Prob[B^{(m-1)} \leq v_i]\big) \\
&\qquad + \textstyle\int_Q^{v_i} (u_i - v_i) \cdot Prob[B^{(m-1)} = \omega] \cdot d\omega + \int_Q^{v_i} (v_i - \omega) \cdot Prob[B^{(m-1)} = \omega] \cdot d\omega = \\
&(u_i - v_i) \cdot \big(Prob[B^{(m)} \leq v_i] - Prob[B^{(m-1)} \leq v_i]\big) \\
&\qquad + (u_i - v_i) \cdot Prob[B^{(m-1)} \leq v_i] + \textstyle\int_Q^{v_i} (v_i - \omega) \cdot \frac{d}{d\omega} Prob[B^{(m-1)} \leq \omega] \cdot d\omega = \\
&(u_i - v_i) \cdot Prob[B^{(m)} \leq v_i] + v_i \cdot \textstyle\int_Q^{v_i} \frac{d}{d\omega} Prob[B^{(m-1)} \leq \omega] \cdot d\omega \\
&\qquad - \textstyle\int_Q^{v_i} \omega \cdot \frac{d}{d\omega} Prob[B^{(m-1)} \leq \omega] \cdot d\omega \Leftrightarrow \\
&EU_i(v_i|\#agents = k) = (u_i - v_i) \cdot \widehat{\Phi}(g^{-1}(v_i)) + v_i \cdot \widehat{Y}(g^{-1}(v_i)) \\
&\qquad - \textstyle\int_Q^{v_i} \omega \cdot \big(\widehat{Y}(g^{-1}(\omega))\big)' \cdot d\omega
\end{aligned}$$

Note that $\int_Q^{v_i} \omega \cdot \big(\widehat{Y}(g^{-1}(\omega))\big)' \cdot d\omega = v_i \cdot \widehat{Y}(g^{-1}(v_i)) - \int_Q^{v_i} (\omega)' \cdot \widehat{Y}(g^{-1}(\omega)) \cdot d\omega$, thus

$$EU_i(v_i|\#agents = k) = (u_i - v_i) \cdot \widehat{\Phi}(g^{-1}(v_i)) + \int_Q^{v_i} \widehat{Y}(g^{-1}(\omega)) \cdot d\omega \tag{8}$$

This equation covers also the case that $k \leq m$, since then it is $\widehat{\Phi}(u) = \widehat{Y}(u) = 1, \forall u \geq Q$.

$EU_i(v_i) = \sum_{k=1}^{N} \pi_k \cdot EU_i(v_i|\#agents = k) \Rightarrow$

$$EU_i(v_i) = (u_i - v_i) \cdot \Phi(g^{-1}(v_i)) + \int_Q^{v_i} Y(g^{-1}(\omega)) \cdot d\omega \tag{9}$$

where $\Phi(x) = \sum_{k=1}^{N} \pi_k \cdot \sum_{i=0}^{m-1} C(k-1, i) \cdot \big(F_Q(x)\big)^{k-1-i} \cdot \big(1 - F_Q(x)\big)^{i} = \sum_{i=0}^{m-1} \sum_{k=i+1}^{N} \pi_k \cdot C(k-1, i) \cdot \big(F_Q(x)\big)^{k-1-i} \cdot \big(1 - F_Q(x)\big)^{i}$

and $Y(x) = \sum_{k=1}^{N} \pi_k \cdot \sum_{i=0}^{m-2} C(k-1, i) \cdot \big(F_Q(x)\big)^{k-1-i} \cdot \big(1 - F_Q(x)\big)^{i} = \sum_{i=0}^{m-2} \sum_{k=i+1}^{N} \pi_k \cdot C(k-1, i) \cdot \big(F_Q(x)\big)^{k-1-i} \cdot \big(1 - F_Q(x)\big)^{i}$

It is $C(k-1, i) = 0$, if $k \leq i$ and this is the reason why we changed the lower bound of

[4] Because from the point of view of a participating agent it does not know whether the other $N-1$ agents participate.

the sum for k from 1 to $i+1$.
We will use the fact that $C(k-1,i)\cdot C(N-1,k-1) = C(N-1-i,N-k)\cdot C(N-1,i)$.
Substituting from equations 6 and 7 and for any $i \in [0, m-1]$ and $x \geq Q$ it is:
$\sum_{k=i+1}^{N} \pi_k \cdot C(k-1,i) \cdot \big(F_Q(x)\big)^{k-1-i} \cdot \big(1-F_Q(x)\big)^{i} =$
$\sum_{k=i+1}^{N} C(k-1,i)\cdot C(N-1,k-1)\cdot (F(Q))^{N-k}\cdot(1-F(Q))^{k-1}\cdot(\frac{F(x)-F(Q)}{1-F(Q)})^{k-1-i}\cdot(\frac{1-F(x)}{1-F(Q)})^{i}$
$= \sum_{k=i+1}^{N} C(N-1-i,N-k)\cdot C(N-1,i)\cdot(F(Q))^{N-k}\cdot(F(x)-F(Q))^{k-1-i}\cdot(1-F(x))^{i}$
$= C(N-1,i)\cdot(1-F(x))^{i}\cdot\sum_{k=i+1}^{N} C(N-1-i,N-k)\cdot(F(Q))^{N-k}\cdot(F(x)-F(Q))^{k-1-i}$
$= C(N-1,i)\cdot(1-F(x))^{i}$
$\quad \cdot \sum_{\lambda=0}^{N-i-1} C(N-1-i,N-i-1-\lambda)\cdot(F(Q))^{N-i-1-\lambda}\cdot(F(x)-F(Q))^{\lambda}$
$= C(N-1,i)\cdot(1-F(x))^{i}\cdot(F(x))^{N-1-i}$
Therefore $\Phi(x) = \sum_{i=0}^{m-1} C(N-1,i)\cdot\big(F(x)\big)^{N-1-i}\cdot\big(1-F(x)\big)^{i}$
and $Y(x) = \sum_{i=0}^{m-2} C(N-1,i)\cdot\big(F(x)\big)^{N-1-i}\cdot\big(1-F(x)\big)^{i}$.
Note if $x < Q$, then $\Phi(x) = Y(x) = 0$ and thus $EU_i(v_i) = 0$.
The bid v_i that maximizes $EU_i(v_i)$ can be found by setting
$\frac{dEU_i(v_i)}{dv_i} = 0 \Leftrightarrow -\Phi(g^{-1}(v_i)) + (u_i - v_i)\cdot\frac{\Phi'(g^{-1}(v_i))}{g'(g^{-1}(v_i))} + Y(g^{-1}(v_i)) = 0$.
Since $v_i = g(u_i)$, the previous equation becomes
$-\Phi(u_i) + \big(u_i - g(u_i)\big)\cdot\frac{\Phi'(u_i)}{g'(u_i)} + Y(u_i) = 0$.
The function $g(u)$ that satisfies this equation is the same as equation 5 (use lemma 6.3 with $T(u) = \Phi(u) - \Psi(u)$) given that the boundary condition is $g(Q) = Q$. ■

Because Φ and Y are $\Phi(x) = \sum_{i=0}^{m-1} C(N-1,i)\cdot\big(F(x)\big)^{N-1-i}\cdot\big(1-F(x)\big)^{i}$ and $Y(x) = \sum_{i=0}^{m-2} C(N-1,i)\cdot\big(F(x)\big)^{N-1-i}\cdot\big(1-F(x)\big)^{i}$ we can simplify equation 5 to:

$$g(u) = u - (F(u))^{-(N-m)}\cdot\int_Q^u (F(z))^{N-m}\cdot dz$$

Note that equation 9 can be written as $EU_i(v_i) = (u_i - v_i)\cdot\Phi(g^{-1}(v_i)) + \int_Q^{g^{-1}(v_i)} Y(u)\cdot g'(u)\cdot du$ and the maximal expected utility $\mathcal{U}(u_i,Q)$ of agent i is therefore
$\mathcal{U}(u_i,Q) = EU_i(v_i)\big|_{v_i=g(u_i)} \Leftrightarrow$

$$\mathcal{U}(u_i,Q) = \big(u_i - g(u_i)\big)\cdot\Phi(u_i) + \int_Q^{u_i} Y(\omega)\cdot g'(\omega)\cdot d\omega \qquad (10)$$

We will use this utility in the next theorems, and in the complete analysis of the full R-round case (when $R \geq 2$), which is presented in [Vetsikas et al. 2007].

THEOREM 4.2. *If the starting price is $Q = 0$, a second round of bidding exists with probability $(1-p)$ ($p \neq 0,1$), the utility of the agents in the second round is drawn from the same distribution $F(u)$ and each agent i in fact has utility in the second round $\widetilde{u}_i \simeq u_i$, then the equilibrium strategy is the solution of the differential equation*

$$\big(u_i - g(u_i)\big)\cdot\frac{\Phi'(u_i)}{g'(u_i)} = \big(\Phi(u_i) - Y(u_i)\big)\cdot\Psi(u_i, g(u_i)) \qquad (11)$$

where
$\Psi(u_i,Q) = 1 + \frac{1-p}{p}\cdot\frac{\Phi(Q)-Y(Q)}{\Phi(u_i)-Y(u_i)}\cdot e^{\int_Q^{u_i}\frac{-Y'(\omega)}{\Phi(\omega)-Y(\omega)}\cdot d\omega}\cdot\big(\Phi(u_i) + \int_Q^{u_i}\frac{Y(\omega)\cdot\Phi'(\omega)}{\Phi(\omega)-Y(\omega)}\cdot d\omega\big)$,
and the boundary condition is $g(0) = 0$.

PROOF. The proof is presented in [Vetsikas et al. 2007].

THEOREM 4.3. *If the starting price is $Q = 0$, a second round of bidding exists with probability $(1-p)$ ($p \neq 0, 1$), the utility of the other agents in the second round is drawn from distribution $H(u)$, and each agent i knows (more accurately for itself) that its own utility $\widetilde{u}_i$ is drawn from distribution $G(u)$, then the equilibrium strategy is the solution of the differential equation*

$$\big(u_i - g(u_i) + \frac{1-p}{p} \cdot U_L(g(u_i))\big) \cdot \frac{\Phi'(u_i)}{g'(u_i)} = \big(\Phi(u_i) - Y(u_i)\big) \cdot \Psi(g(u_i)) \qquad (12)$$

where $\Psi(Q)$ is given by equation 15 and $U_L(Q)$ by equation 13.
The boundary condition is $g(0) = 0$.

PROOF. Initially we must compute the expected gain of utility (actually it's negative, so it's a loss) $U_L(Q)$, if the agent is a winner in the first round and in the second his utility $\widetilde{u}_i < Q$. The agent is forced (by the rules) to keep a bid of at least Q in the auction; so it puts a bid $\widetilde{v}_i = Q$. Let us assume that exactly k of the other agents wish to buy the goods. If $k \geq m$ there is no problem, and the utility difference is 0. However if $k < m$, there are $(m-k)$ units that must be sold randomly to some of the previous winners. We can compute that the probability of getting a room in this case is equal to $\frac{N\cdot(m-k)}{m\cdot(N-k)}$.[5] The probability that k of the other agents will have utilities $\widetilde{u}_j \geq Q$ is $C(N-1,k) \cdot \big(H(Q)\big)^{N-1-k} \cdot \big(1-H(Q)\big)^k$ (see lemma 6.2) and, if selected, the utility difference is $\widetilde{u}_i - Q$. Thus the expected utility difference $U_L(\widetilde{u}_i, Q)$ is

$$U_L(\widetilde{u}_i, Q) = \sum_{k=0}^{m-1} \Big\{ \frac{N\cdot(m-k)}{m\cdot(N-k)} \cdot C(N-1,k) \cdot \big(H(Q)\big)^{N-1-k} \cdot \big(1-H(Q)\big)^k \cdot (\widetilde{u}_i - Q) \Big\}$$

if $\widetilde{u}_i < Q$.
If $\widetilde{u}_i \geq Q$, then we can't have this case, since the agent would wish to participate. Since the value of the utility $\widetilde{u}_i$ is not known, but the distribution $G(u)$, from which it is drawn, is the total utility difference (loss actually), because of forced bidding of the agents that won the first round, is $U_L(Q) = \int_0^Q U_L(\widetilde{u}_i, Q) \cdot Prob[\widetilde{u}_i = \omega] \cdot d\omega$.
Note also that $\int_0^Q (\widetilde{u}_i - Q) \cdot Prob[\widetilde{u}_i = \omega] \cdot d\omega = \int_0^Q G(\omega) \cdot d\omega$.
It is therefore

$$U_L(Q) = -\sum_{k=0}^{m-1} \Big\{ \frac{N\cdot(m-k)}{m\cdot(N-k)} \cdot C(N-1,k) \cdot \big(H(Q)\big)^{N-1-k} \cdot \big(1-H(Q)\big)^k \cdot \int_0^Q G(\omega) \cdot d\omega \Big\} \qquad (13)$$

The expected utility for the agent at the second round (without the inclusion of the utility difference stated in equation 13) is $\widetilde{U}(Q) = \int_Q^{+\infty} \mathcal{U}(\omega, Q) \cdot Prob[\widetilde{u}_i = \omega] \cdot d\omega$. This is because, if $\widetilde{u}_i < Q$, then the utility gained is 0. If $\widetilde{u}_i \geq Q$, then the utility gained is $\mathcal{U}(\omega, Q)$, and it is computed by equations 5 and 10, in which $\Phi(x)$ and $Y(x)$ are replaced by $\widetilde{\Phi}(x) = \sum_{i=0}^{m-1} C(N-1,i) \cdot \big(H(x)\big)^{N-1-i} \cdot \big(1-H(x)\big)^i$ and
$\widetilde{Y}(x) = \sum_{i=0}^{m-2} C(N-1,i) \cdot \big(H(x)\big)^{N-1-i} \cdot \big(1-H(x)\big)^i$.
Both utilities $\widetilde{U}(Q)$ and $U_L(Q)$ depend on the price Q at the start of the second round, which in turn depends on the bids placed in the first round.
If $B^{(m)} > v_i$ then $Q = B^{(m)}$; in this case the agent does not win in the first round, so there

[5] This is bigger than $\frac{m-k}{m}$, because some of the k agents might have been winners on the first round as well.

is no utility loss in the second round. If $B^{(m-1)} > v_i \geq B^{(m)}$, then the agent submitted the m^{th} price, so $Q = v_i$, and if $v_i \geq B^{(m-1)}$, then $Q = B^{(m-1)}$; in both these cases there is an expected loss in the second round.
Note that $Prob[B^{(m-1)} \leqslant v] = Y(g^{-1}(v))$ and $Prob[B^{(m)} \leqslant v] = \Phi(g^{-1}(v))$ (see proof of theorem 4.1), and that $Prob[Q = v_i] = Prob[B^{(m-1)} > v_i \geq B^{(m)}] = Prob[B^{(m)} \leq v_i] - Prob[B^{(m-1)} \leq v_i] = \Phi(g^{-1}(v)) - Y(g^{-1}(v))$.
As a result we can now compute the expected utility, if a second round does exist, $U_i^{(2)}$ as follows:

$$U_i^{(2)} = \int_0^{u_i} \mathcal{U}(\omega) \cdot Prob[Q = \omega] \cdot d\omega =$$
$$\int_0^{v_i^-} \big(\widetilde{U}(\omega) + U_L(\omega)\big) \cdot Prob[Q = \omega] \cdot d\omega + \big(\widetilde{U}(v_i) + U_L(v_i)\big) \cdot Prob[Q = v_i]$$
$$+ \int_{v_i+}^{u_i} \widetilde{U}(\omega) \cdot Prob[Q = \omega] \cdot d\omega \Leftrightarrow$$
$$U_i^{(2)} = \int_0^{v_i} \big(\widetilde{U}(\omega) + U_L(\omega)\big) \cdot \tfrac{d}{d\omega} Y(g^{-1}(\omega)) \cdot d\omega$$
$$+ \big(\widetilde{U}(v_i) + U_L(v_i)\big) \cdot \big\{\Phi(g^{-1}(v_i)) - Y(g^{-1}(v_i))\big\} + \int_{v_i}^{u_i} \widetilde{U}(\omega) \cdot \tfrac{d}{d\omega}\Phi(g^{-1}(\omega)) \cdot d\omega$$

The expected utility if the auction closes at the first round $U_i^{(1)}$ is given by equation 9 when $Q = 0$ is inserted, since the price is $Q = 0$ at the beginning of the first round.
The expected utility for both rounds is $EU_i(v_i) = p \cdot U_i^{(1)} + (1 - p) \cdot U_i^{(2)} \Leftrightarrow$

$$EU_i(v_i) = p \cdot \big\{(u_i - v_i) \cdot \Phi(g^{-1}(v_i)) + \int_0^{v_i} Y(g^{-1}(\omega)) \cdot d\omega\big\}$$
$$+(1 - p) \cdot \Big\{ \int_0^{v_i} \big(\widetilde{U}(\omega) + U_L(\omega)\big) \cdot \tfrac{d}{d\omega} Y(g^{-1}(\omega)) \cdot d\omega \qquad (14)$$
$$+ \big(\widetilde{U}(v_i) + U_L(v_i)\big) \cdot \big\{\Phi(g^{-1}(v_i)) - Y(g^{-1}(v_i))\big\} + \int_{v_i}^{u_i} \widetilde{U}(\omega) \cdot \tfrac{d}{d\omega}\Phi(g^{-1}(\omega)) \cdot d\omega\Big\}$$

We find the equilibrium by setting $\frac{dU_i(v_i)}{dv_i} = 0$ and then substituting $v_i = g(u_i)$. In the end we get

$$(u_i - g(u_i) + \frac{1-p}{p} \cdot U_L(g(u_i))) \cdot \frac{\Phi'(u_i)}{g'(u_i)} = (1 - \frac{1-p}{p} \cdot (\widetilde{U}'(g(u_i)) + U_L'(g(u_i)))) \cdot (\Phi(u_i) - Y(u_i))$$

Let us set $\Psi(Q) = 1 - \frac{1-p}{p} \cdot (\widetilde{U}'(Q) + U_L'(Q))$, therefore

$$\Psi(Q) = 1 + \tfrac{1-p}{p} \cdot \Big\{ \int_Q^{+\infty} G'(z) \cdot \tfrac{\widetilde{\Phi}(Q) - \widetilde{Y}(Q)}{\widetilde{\Phi}(z) - \widetilde{Y}(z)} \cdot e^{\int_Q^z \frac{-\widetilde{Y}'(\omega)}{\widetilde{\Phi}(\omega) - \widetilde{Y}(\omega)} \cdot d\omega} \cdot (\widetilde{\Phi}(z) + \int_Q^z \tfrac{\widetilde{Y}(\omega) \cdot \widetilde{\Phi}'(\omega)}{\widetilde{\Phi}(\omega) - \widetilde{Y}(\omega)} \cdot d\omega) \cdot dz +$$
$$\sum_{k=0}^{m-1} \tfrac{N \cdot (m-k)}{m \cdot (N-k)} \cdot C(N-1, k) \cdot \{(N-1-k) \cdot H'(Q) \cdot \big(H(Q)\big)^{N-2-k} \cdot \big(1 - H(Q)\big)^k \cdot \int_0^Q G(\omega) \cdot d\omega$$
$$-k \cdot H'(Q) \cdot \big(H(Q)\big)^{N-1-k} \cdot \big(1 - H(Q)\big)^{(k-1)} \cdot \int_0^Q G(\omega) \cdot d\omega$$
$$+\big(H(Q)\big)^{N-1-k} \cdot \big(1 - H(Q)\big)^k \cdot G(Q)\}\Big\} \qquad (15)$$

where $\widetilde{\Phi}(x) = \sum_{i=0}^{m-1} C(N-1, i) \cdot (H(x))^{N-1-i} \cdot (1 - H(x))^i$ and $\widetilde{Y}(x) = \sum_{i=0}^{m-2} C(N-1, i) \cdot (H(x))^{N-1-i} \cdot (1 - H(x))^i$. Then the differential equation becomes

$$(u_i - g(u_i) + \frac{1-p}{p} \cdot U_L(g(u_i))) \cdot \frac{\Phi'(u_i)}{g'(u_i)} = (\Phi(u_i) - Y(u_i)) \cdot \Psi(g(u_i))$$

The boundary condition is $g(0) = 0$. ■

5. DISCUSSION AND CONCLUSIONS

To summarize, in this paper we concentrated on auctions that have a set of possible closing times, one of which is chosen randomly for the auction to end at. These auctions can be decomposed into one or more rounds, during which the auction is treated as sealed bid. We analyzed the Bayes-Nash equilibria that exist in such cases and computed several "novel" equilibria for these auctions. We computed equilibria for auctions that sell 1 or several identical items, and we demonstrated the methodology used to compute the equilibria in these cases.

In [Vetsikas et al. 2007], we use the same methodology that we applied in sections 3 and 4, in order to derive the systems of differential equations whose solution constitutes a Bayes-Nash equilibrium for the general case of an auction with multiple possible closing times, and not just two, as was the case in the theorems that were presented here. We break the auction into more rounds, compute the utility for the last two rounds (e.g. solving equation 12 and then substituting in equation 14), and then recursively compute the expected utilities for the previous rounds, using the last utility function as the next round's utility function, until the first round is reached. In addition we describe the algorithm used in order to solve this system, and present equilibria both for a uniform distribution and a distribution that approximates the values of hotel rooms during the TAC game.[6] We use this later equilibrium strategy in our TAC agent.

We are currently working towards extending the results presented here and in [Vetsikas et al. 2007], in various ways. We examine the equilibria present in the case of multi-demand auctions; this would allow us to remove the current restriction that each agent must bid for a single item. In addition, we are interested in the change that occurs in the equilibrium strategies, when each agent wishes not only to maximize its own profit, but also to minimize the profit of its opponents, which is more realistic for a competition setting, like TAC. We plan to incorporate these extensions in our analysis in order to further improve our TAC agents.

6. APPENDIX

LEMMA 6.1. *If random variables $X_i, \forall i \in \{1, \ldots, N\}$ are i.i.d. with probability distribution $f(x) = Prob[X_i \leqslant x]$ and $Y_N^{(k)}$ denotes the k^{th} order statistic of the variables X_i, then*

$$Prob[Y_N^{(k)} \leqslant y] = \sum_{i=0}^{k-1} C(N,i) \cdot (f(y))^{N-i} \cdot (1 - f(y))^i$$

where $C(N,i)$ is the total number of possible combinations of i items chosen from N.

LEMMA 6.2. *If random variables $X_i, \forall i \in \{1, \ldots, N\}$ are i.i.d. with probability distribution $f(x) = Prob[X_i \leqslant x]$, then the probability p_k that exactly k of these variables $X_i \geq T$ is*

$$p_k = C(N,k) \cdot (f(T))^{N-k} \cdot (1 - f(T))^k$$

PROOF. Both of these lemmas can be found in any probabilities book (e.g. [Rice 1995]).

LEMMA 6.3. *A function $g(u)$ that satisfies the equation $(u - g(u)) \cdot \frac{\Phi'(u)}{g'(u)} = T(u)$ is the following*

$$g(u) = u - \frac{\Omega(u)}{T(u)} \cdot \int_C^u \frac{T(\omega)}{\Omega(\omega)} \cdot d\omega$$

where $\Omega(u) = e^{\int_D^u \frac{T'(\omega) - \Phi'(\omega)}{T(\omega)} \cdot d\omega}$, C depends on the boundary conditions and D can have any value.

PROOF. All that is needed to do is to replace $g(u)$ in the differential equation with the formula provided and check that the two sides of the equation are indeed equal.

[6]This distribution is generated by collecting the utilities of the hotel rooms from a large number of actual games and thus we can estimate the distributions $F(u), G(u), H(u)$ that we use in the formulas for the equilibria.

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