Implementable Allocation Rules

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1. BACKGROUND

Let A be a finite set of goods. R^A is the set of all valuation functions on A, that is the set of all real valued functions defined on A. The value of a for a buyer with valuation v is thus v_a. It is convenient to represent each good a by its associated unit vector $e^a \in R^A$, where $e^a_a = 1$ and $e^a_b = 0$ for every $b \neq a$. Let $Z(A)$ be the set of all sub-probability vectors $z \in R^A$. That is, $Z(A) = \{z \in R^A |\ z^a \geq 0\ \forall a,\ \sum_{a \in A} z^a \leq 1\}$. Let $D \subseteq R^A$, and let $f : D \to Z(A)$. We think of D as the set of all possible valuations of a given buyer with quasi-linear utility function, and f is interpreted as a *randomized allocation rule* in some direct mechanism (D, f, c), where $c : D \to R$; If a buyer with valuation v declares w she receives a with probability $f^a(w)$, and therefore she evaluates $f(w)$ by the inner product, $\langle v, f(w)\rangle = \sum_{a \in A} v_a f^a(w)$ minus $c(w)$. If $\sum_{a \in A} f^a(w) < 1$, there is a positive probability that the buyer does not receive any good. In such case the utility of the outside option is assumed to equal zero. A randomized allocation rule satisfying $f(v) \in \{e^a | a \in A\}$ for every $v \in D$ is called in this manuscript a *pure allocation* rule. A randomized allocation rule f is *finite-valued* if its range $\{f(v)|v \in D\}$ is a finite set.

We say that a randomized allocation rule f is *implementable* if there exists a function $c : D \to R$ such that truth telling is a dominant strategy in the direct mechanism (D, f, c). That is,

$$\langle v, f(v)\rangle - c(v) \geq \langle v, f(w)\rangle - c(w) \quad \forall v, w \in D. \tag{1}$$

If f is implementable, then a simple manipulation on (1) shows that:

$$\langle f(v) - f(w), v - w\rangle \geq 0 \quad \text{ for every } v, w \in D. \tag{2}$$

A randomized allocation rule satisfying (2) is called *monotone*. What we call monotone is called weakly monotone by [Bikhchandani et al. 2006] and others. However, the term "monotone" is well-known in the convex analysis literature (e.g., [Rockafellar 1970]) and we therefore use it. By further using (1) it was noticed by [Rochet 1987] that every implementable randomized allocation rule f satisfies a stronger

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monotonicity property; f is called *cyclically monotone* if for every $k \geq 2$, for every k vectors in D (not necessarily distinct), $v_1, v_2, \ldots, v_k$ the following holds:

$$\sum_{i=1}^{k} \langle v_i - v_{i+1}, f(v_i) \rangle \geq 0, \tag{3}$$

where v_{k+1} is defined to be v_1. By taking $k = 2$ in (3) it can be easily seen that every cyclically monotone randomized allocation rule is monotone. The following characterization of implementability is known:

Theorem(Rochet 1987): *A randomized allocation rule is implementable if and only if it is cyclically monotone.*

However, It is well known that not every monotone randomized allocation rule is cyclically monotone; An example can be easily created from a related example in [Rockafellar 1970]. The following intriguing question has been posed in [Bikhchandani et al. 2006]:

Question(Bikhchandani, Chatterji, Lavi, Mu'alem, Nisan, and Sen 2006):
For which domains D is every pure monotone allocation rule on D is implementable?

The answer to this question may be important in the area of mechanism design for anyone looking for a concrete characterization of (approximated) implementable allocation rules for a particular domain that steams out of a given real life problem – it just tell her where to look for such mechanisms; See e.g., [Koutsoupias and Vidali 2007; Ashlagi 2008]. Mathematical curiosity is also a good reason to deal with this question. In [Bikhchandani et al. 2006] it was proved that $D = R_+^A$ as well as many other domains, all of them convex, are such domains. In [Gui et al. 2004] it was noticed that by a result of Roberts, [Roberts 1979] the full domain $D = R^A$ is such a domain, and it was proved in addition that so is every cube. Finally, in [Saks and Yu 2005] it was proved that every convex domain satisfies this requirement:

Theorem(Saks and Yu): *Every pure monotone allocation rule on a convex domain is implementable.*

We say that a domain of valuation functions is a *monotonicity domain* if every finite-valued monotone randomized allocation rule defined on it is implementable. It is well known (and was noticed for the Bayesian setting in [Myerson 1981]) that every domain of dimension at most one is a monotonicity domain. As a final important comment we note that in the context of the problems discussed in this paper, there is no loss of generality in dealing with one buyer. This follows from the fact that in a multi-buyers model, a randomized allocation rule is implementable if and only if for each buyer, for each fixed vector of valuations of all other buyers, the resulting one-buyer randomized allocation rule is implementable.

2. OUR RESULTS

In [Monderer 2008] we prove:

Theorem: *Every domain with a convex closure is a monotonicity domain.*

This theorem implies the theorem of Saks and Yu, but with some efforts, it can be directly derived from their theorem. However, to our opinion our proof is significantly simpler than the elaborated proof of Saks and Yu. We further prove:

Theorem: *The closure of every monotonicity domain of dimension 2 is convex.*

3. REMARKS AND FUTURE RESEARCH

It is not known to us whether the conclusions of the above theorem hold for dimensions $d \geq 3$. It can be seen that the converse to Saks and Yu's theorem does not hold. That is, there exist non-convex domains for which every monotone pure allocation rule is implementable. Such an example is given in [Mu'alem and Schapira 2008]. Hence, one may wish to look for non-convex domains for which every pure allocation rule is implementable, which is indeed done in [Mu'alem and Schapira 2008]. An additional research direction in this area would be trying to obtain other sufficient and/or necessary conditions (not monotonicity or cyclic monotonicity) for implementability. Such an approach is taken in [Archer and Kleinberg 2008]. We consider the solution of problems related to monotonicity to be a tool for solving the very important question of finding concrete characterizations of implementable allocation rules such as the one appearing at the famous paper [Roberts 1979], where it was proved that every implementable allocation rule is an affine maximizer (See [Lavi et al. 2003] for an example of newer results in this direction). Finally, it is very important to explore the topic of implementable allocation rules in the Bayesian settings.

REFERENCES

ARCHER, A. AND KLEINBERG, R. 2008. Truthful games are contagious: a local-to-global characterization of truthfulness. In *Proceedings of the 9th ACM Conference on Electronic Commerce.*

ASHLAGI, I. 2008. A characterization of anonymous truth-revealing position auctions. Working Paper, Technion – Israel Institute of Technology; `http://www.technion.ac.il/~ashlagii/papers/TPEC2.pdf`.

BIKHCHANDANI, S., CHATTERJI, S., LAVI, R., MU'ALEM, A., NISAN, N., AND SEN, A. 2006. Weak monotonicity characterizes deterministic dominant strategy implementation. *Econometrica 74(4)*, 1109–1132.

GUI, H., MÜLLER, R., AND VOHRA, R. 2004. Dominant strategy mechanisms with multidimensional types. Northwestern Universioty, Working Paper.

KOUTSOUPIAS, E. AND VIDALI, A. 2007. A Lower Bound of 1+phi for Truthful Scheduling Mechanisms. In *MFCS*. 454–464.

LAVI, R., MU'ALEM, A., AND NISAN, N. 2003. Towards a characterization of truthful combinatorial auctions. In *FOCS'03.*

MONDERER, D. 2008. Monotonicity and implementability. Working Paper, Technion – Israel Institute of Technology; `http://ie.technion.ac.il/~dov/monderer07_monotone.pdf`; A one-page anstract appears at the Proceedings of the 9th ACM Conference on Electronic Commerce.

MU'ALEM, A. AND SCHAPIRA, M. 2008. Mechanism design over discrete domains. In *Proceedings of the 9th ACM Conference on Electronic Commerce.*

MYERSON, R. B. 1981. Optimal Auction Design. *Mathematics of Operations Research (6) 21*, 58–73.

ROBERTS, K. 1979. The characterization of implementable choice rules. In *Aggregation and Revelation of Preferences*, J.-J. Laffont, Ed. North Holland Publishing Company.

ROCHET, J. 1987. A necessary and sufficient condition for rationalizability in a quasi-linear context. *Journal of Mathematical Economics 16*, 191–200.

ROCKAFELLAR, R. T. 1970. *Convex Analysis.* Princeton University Press.

SAKS, M. AND YU, L. 2005. Weak monotonicity suffices for truthfulness on convex domains. In *Proceedings of the 6th ACM conference on Electronic commerce.* 286–293.