Solution to Exchanges 7.3 Puzzle: Product Adoption in a Social Network

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Two correct solutions were submitted to the puzzle in SIGecom exchanges given at http://www.sigecom.org/exchanges/volume_7/3/PUZZLE.pdf. Both of these solutions are listed below. The first is by Aneesh Sharma, the second by Sicco Verwer.

Solution 1
Let $B_i$ denote the expected number of $B$ adoptors after $i$ agents have made their adoption decisions. We are interested in computing $B_n/n$. First, we observe that $B_1 = p_0$ as the first agent can only adopt $B$ if she is a $B$ fanatic. Further, we observe that for any $i > 1$:

$$B_{i+1} = \left( p_0 + \frac{B_i}{n} p_1 \right) (B_i + 1) + \left( 1 - p_0 - \frac{B_i}{n} p_1 \right) B_i$$

This is because the expected number of agents go up by 1 only if either agent $i+1$ is a $B$ fanatic or if the agent that $i+1$ has chosen to admire has already chosen to adopt $B$ (with probability $B_i/n$). In the remaining cases, the expected number of agents remain the same. Now, we can simplify the above equation to get:

$$B_{i+1} = p_0 + \left( 1 + \frac{p_1}{n} \right) B_i$$

We can telescope this sum starting with $B_1$ to get:

$$B_{n+1} = p_0 \left( 1 + \left( 1 + \frac{p_1}{n} \right) + \ldots + \left( 1 + \frac{p_1}{n} \right)^{n} \right)$$

Summing the series for $i = n - 1$ and using the approximation $(1 + z/n)^n \approx e^z$ for large $n$, we have the quantity of interest as:

$$\frac{B_n}{n} \approx \frac{p_0}{p_1} (e^{p_1} - 1)$$

Solution 2
Like the previous solution, we obtain:

$$B_{i+1} = p_0 + \left( 1 + \frac{p_1}{n} \right) B_i$$

However, we write out this sum as:

$$B_n = p_0 \left( n + a_n \cdot \frac{p_1}{n} + b_n \left( \frac{p_1}{n} \right)^2 + c_n \left( \frac{p_1}{n} \right)^3 \ldots \right)$$

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Now we only need to find the values for $a_n, b_n, c_n, \ldots$. Here $a_n$ stands for the number of times that $p_0 \cdot p_0 / n$ occurs in $B_n$. This is equal to the number of times that $p_0 \cdot p_0 / n$ occurs in $B_{n-1}$ plus the number of times that $p_0$ occurs in $B_{n-1}$. Thus:

$$a_n = a_{n-1} + (n - 1)$$

Similarly for $b_n, c_n, \ldots$:

$$b_n = b_{n-1} + a_{n-1}$$
$$c_n = c_{n-1} + b_{n-1}$$

\[ \ldots \]

Except the first few values, these recursions form the triangular, tetrahedral, pentatopic, etc. numbers. Solving these recursions results in the following sets of equations:

$$a_n = \frac{1}{2} n (n + 1)$$
$$b_n = \frac{1}{6} n (n + 1) (n + 2)$$
$$c_n = \frac{1}{24} n (n + 1) (n + 2) (n + 3)$$

\[ \ldots \]

For $n$ goes to $\infty$, these numbers can be used to rewrite the final result $B_n / n$ as:

$$\frac{B_n}{n} \approx p_0 \left( 1 + \frac{p_1}{2} + \frac{p_1^2}{6} + \frac{p_1^3}{24} \ldots \right)$$

The sequence $f(x) = 2, 6, 24, \ldots$ is equal to $f(x) = (x + 2)!$, thus:

$$\frac{B_n}{n} \approx p_0 \left( 1 + \sum_{i=2}^{\infty} \frac{p_1^{i-1}}{i!} \right)$$

Using $e^x = \sum_{i=0}^{\infty} (x^n / n!)$, we obtain:

$$\frac{B_n}{n} \approx p_0 \left( 1 + \frac{1}{p_1} \sum_{i=2}^{\infty} \frac{p_1^i}{i!} \right)$$
$$\frac{B_n}{n} \approx p_0 \left( 1 + \frac{1}{p_1} \left( \sum_{i=0}^{\infty} \frac{p_1^i}{i!} - 1 - p_1 \right) \right)$$
$$\frac{B_n}{n} \approx p_0 \left( 1 + \frac{1}{p_1} (e^{p_1} - 1 - p_1) \right)$$
$$\frac{B_n}{n} \approx \frac{p_0}{p_1} (e^{p_1} - 1)$$