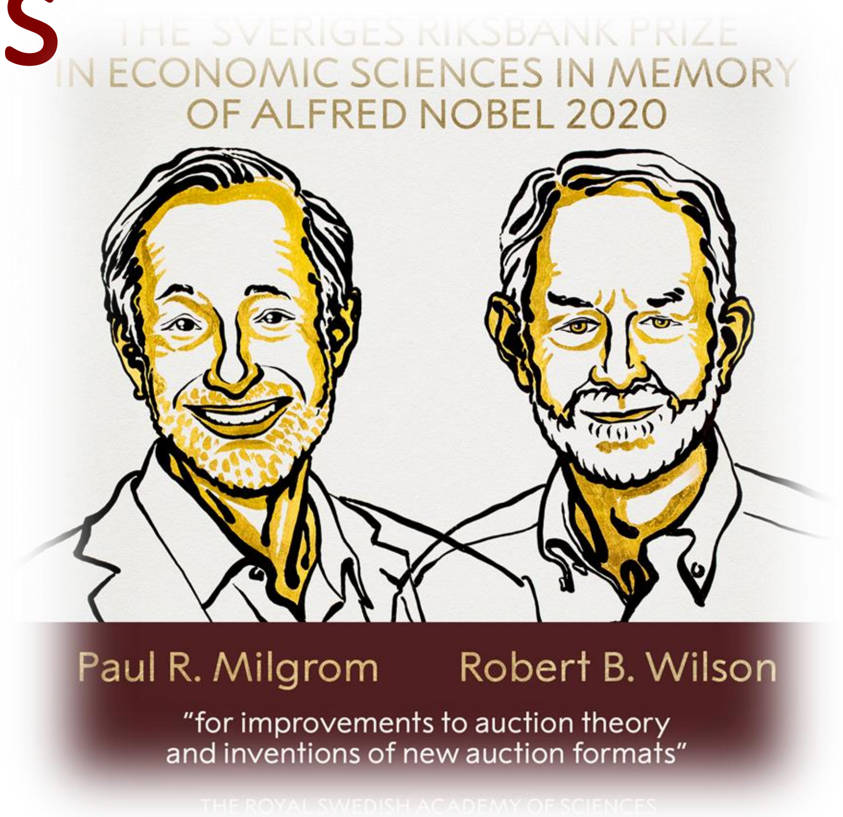
Auctions with Correlated and Interdependent Values

Part I: Contributions of Robert Wilson and Paul Milgrom



Presented by Inbal Talgam-Cohen (Technion)
Inaugural SIGecom Winter Meeting

Foundational Papers We'll Touch Upon

Analysis of leading auction formats (1^{st} price, 2^{nd} price, English)

1. Wilson'77, "A bidding model of perfect competition", The Review of Economic Studies
2. Milgrom'81, "Rational expectations, information acquisition, and competitive bidding", Econometrica
3. Milgrom-Weber'82, "A theory of auctions and competitive bidding", Econometrica

Auction design and robustness

4. Cremer-McLean'88, "Full extraction of the surplus in Bayesian and dominant strategy auctions", Econometrica
5. Wilson'87, "Game-theoretic analyses of trading processes", Advances in Economic Theory: Fifth World Congress

Let's Revisit the Common-Value Setting

Setting:

- Symmetric bidders
- Every bidder i:
  - has value $u_i(S, \boldsymbol{X}) = S$
  - knows only her own signal X_i
- Signal distribution is known

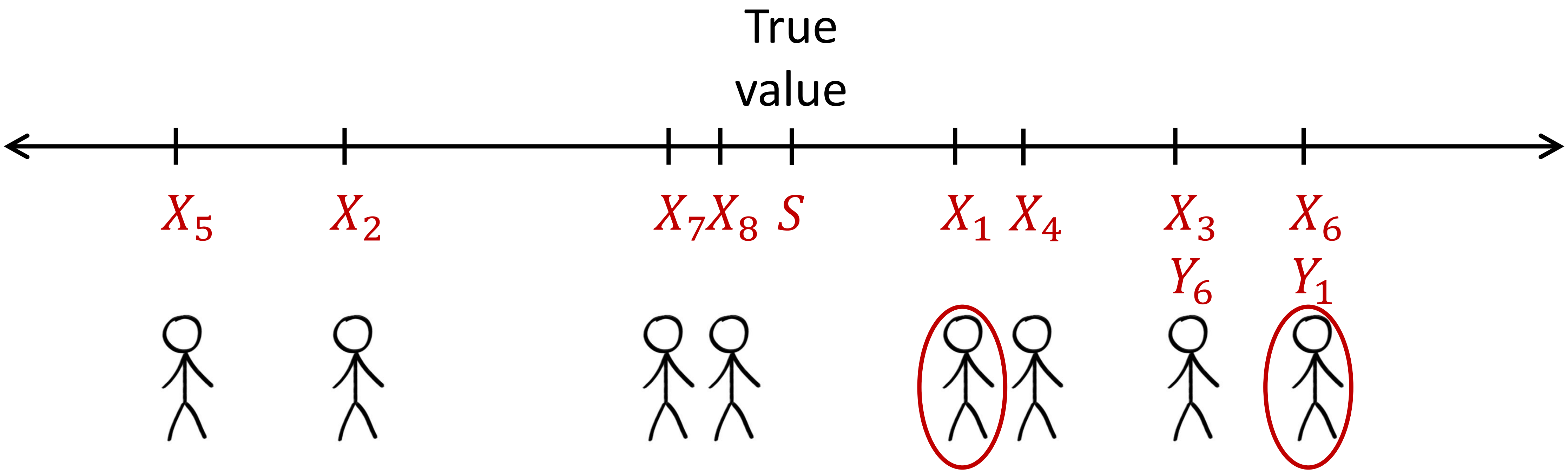
Notation:

- Y_i = highest signal of bidders other than i

Value S

Common-Value Setting

- For concreteness let $X_i = S + \epsilon_i$ where ϵ_{i}'s are i.i.d. and $\mathbb{E}[\epsilon_i] = 0$



Three Values of Interest

- Initial estimate of S by bidder i:

$$\hat{S}_i = \mathbb{E}[S \mid X_i\,]$$

- Estimate of S upon winning the auction:

$$\widehat{W}_i = \mathbb{E}[S \mid X_i, i \text{ won, everything learned during auction}]$$

- Bidder i's bid $\beta(x)$
  - where bidding strategy $\beta(\cdot)$ is symmetric, increasing

$\hat{S}_i$ = value estimate
$\widehat{W}_i$ = estimate upon win
$\beta(x)$ = bid

Is Winning Good or Bad News?

Let's first consider 1st price auctions, with bids according to strategy β

- A priori, i's estimate $\hat{S}_i$ of S is simply $X_i = S + \epsilon_i$, an unbiased estimator:

$$\mathbb{E}[X_i \mid S = s] = s$$

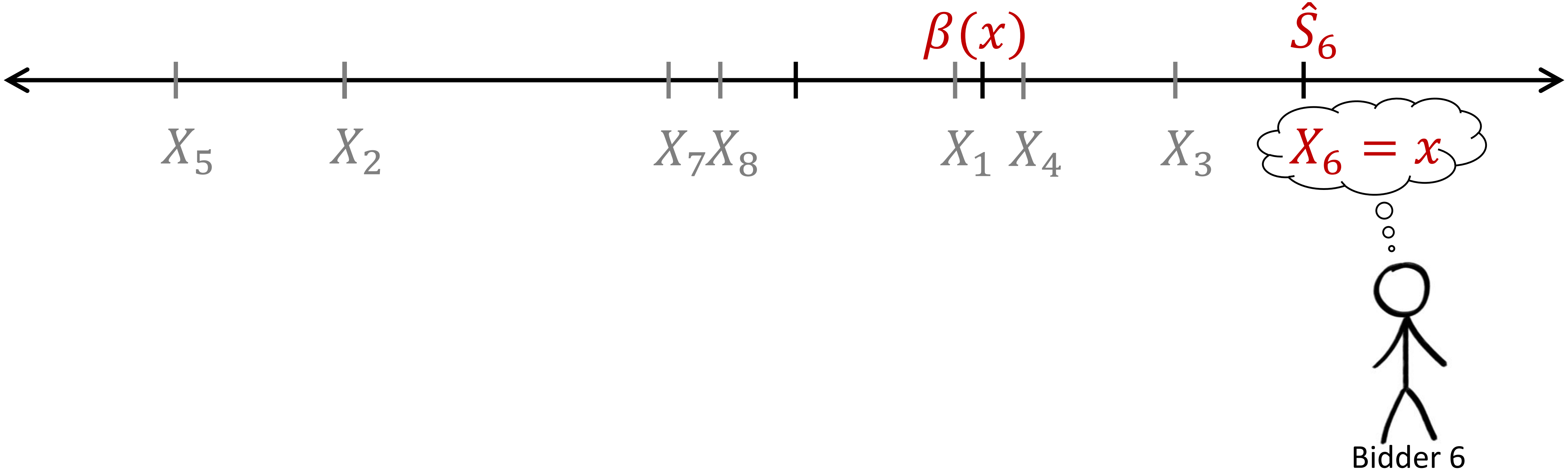
- Upon winning, i learns $X_i = \max_{i'}\{X_{i'}\}$, but now:

$$\mathbb{E}\left[\max_{i'}\{X_{i'}\} \mid S = s\right] > s$$

<u>Conclusion</u>: $\widehat{W}_i < \hat{S}_i$, i.e., winning is bad news

$\hat{S}_i$ = value estimate
$\widehat{W}_i$ = estimate upon win
$\beta(x)$ = bid

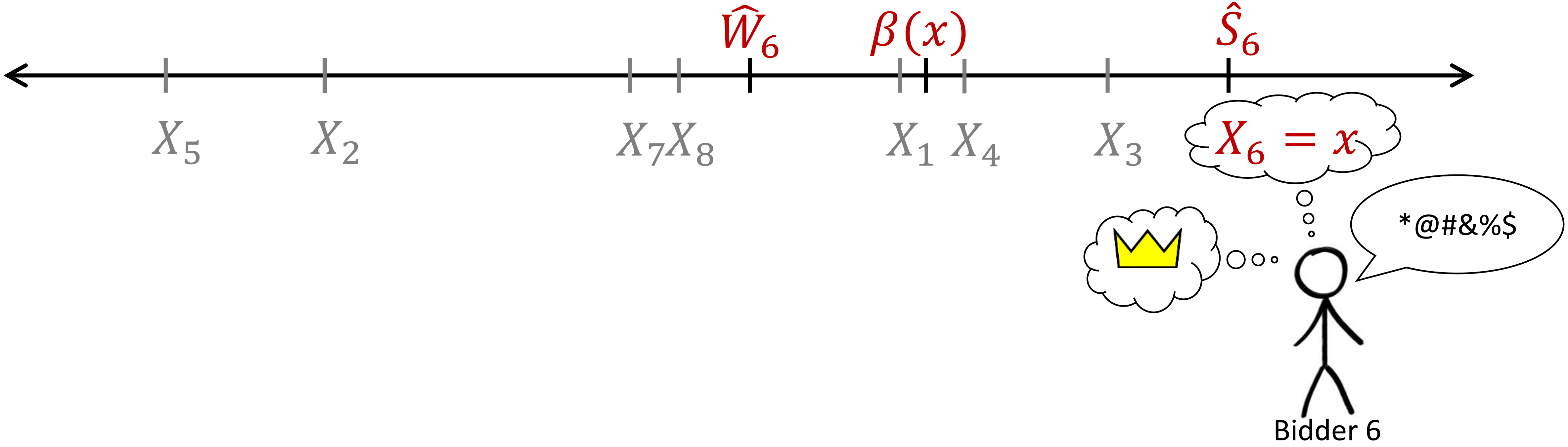
Winner's Curse



$\hat{S}_i$ = value estimate
$\widehat{W}_i$ = estimate upon win
$\beta(x)$ = bid

Winner's Curse

- Winner's curse occurs when $\beta(x)$ does not anticipate the shift from $\hat{S}_i$ to $\widehat{W}_i$



Winner's Curse: Implication to 1st Price

- Let $n \to \infty$
  - Enough information on the market to estimate S accurately

- But if the winner's curse occurs, the price converges to something higher than S

Conclusion: Winner's curse $\Longrightarrow$ price fails to aggregate information

Does winner's curse occur in equilibrium?

$\widehat{W}_i$ = estimate upon win
$\beta(x)$ = bid

Contribution of [Wilson'77]

- Derives closed-form equilibrium bidding strategy $\beta(\cdot)$
  - $\beta(\cdot)$ is symmetric, increasing under appropriate regularity conditions (affiliation)

- In equilibrium, bidders shade their bids to avoid the winner's curse:

$$\beta(x) \leq \widehat{W}_i = \mathbb{E}[S \mid X_i = x, \underbrace{Y_i \leq x}_{X_i \text{ is highest}}]$$

<u>Conclusion</u>: No winner's curse in equilibrium; price converges to true value!

$\widehat{W}_i =$ estimate upon win
$\beta(x) =$ bid

What about 2nd Price Auctions?

- Insight from 1st price analysis:

Equilibrium bids avoid the winner's curse by factoring in the anticipated learning from winning

- Let's use this insight to find equilibrium $\beta(\cdot)$ for 2nd price

$\widehat{W}_i$ = estimate upon win
$\beta(x)$ = bid

Learning from Winning in 2nd Price

- Consider 2nd price auctions, with bids according to strategy β

- What does i learn if she wins?
  - $X_i \geq Y_i$
  - $\beta(Y_i)$

- So $\widehat{W}_i = \mathbb{E}[S \mid X_i = x, Y_i = y]$
- (Cf., $\widehat{W}_i = \mathbb{E}[S \mid X_i = x, Y_i \leq x]$ in 1st price)

$\widehat{W}_i$ = estimate upon win
$\beta(x)$ = bid

Equilibrium Strategy [Milgrom'81]

- No knowledge of $\widehat{W}_i = \mathbb{E}[S \mid X_i = x, Y_i = y]$ at bid time…

Theorem: In 2nd price auctions, under regularity conditions, the following bidding strategy is a symmetric increasing equilibrium:

$$\beta(x) = \mathbb{E}[S \mid X_i = x, Y_i = x]$$

- Indeed, winner i has no incentive to change bid after learning price:

$$\widehat{W}_i = \mathbb{E}[S \mid X_i = x, Y_i = y] \underset{\text{Regularity/affiliation}}{\geq} \underbrace{\mathbb{E}[S \mid X_i = y, Y_i = y]}_{\text{Price}} = \beta(y)$$

Affiliated Signals

If some signals are high, the remaining signals are more likely to be high

Def: Random variables $X_1, \dots, X_n$ with joint density f are affiliated if for every two realizations $x, y \in \mathbb{R}^n$,

$$f(x)f(y) \le f(\underbrace{x \wedge y}_{\text{Component -wise min}})f(\underbrace{x \vee y}_{\text{Component -wise max}})$$

- Equivalent to log-supermodularity of f
- E.g., $f(2,1,8)f(9,7,1) \le f(2,1,1)f(9,7,8)$

Affiliation Implies Increasing Bid $\beta(\cdot)$

$$\beta(x) = \mathbb{E}[S \mid X_i = Y_i = x]$$

S, X_i, Y_i affiliated

- **Intuition**: Affiliation means positive correlation everywhere

Linkage Principle [Milgrom-Weber'82]

Overview of Contributions

Consider a general symmetric model:

$$u_i(S, \boldsymbol{X}) = u(X_i, \{X_j\}_{j \neq i}) \text{ for all } i$$

Regularity assumption: affiliation

Results:

1. Symmetric increasing equilibria for English, 1st, 2nd price auctions
2. Revenue hierarchy:

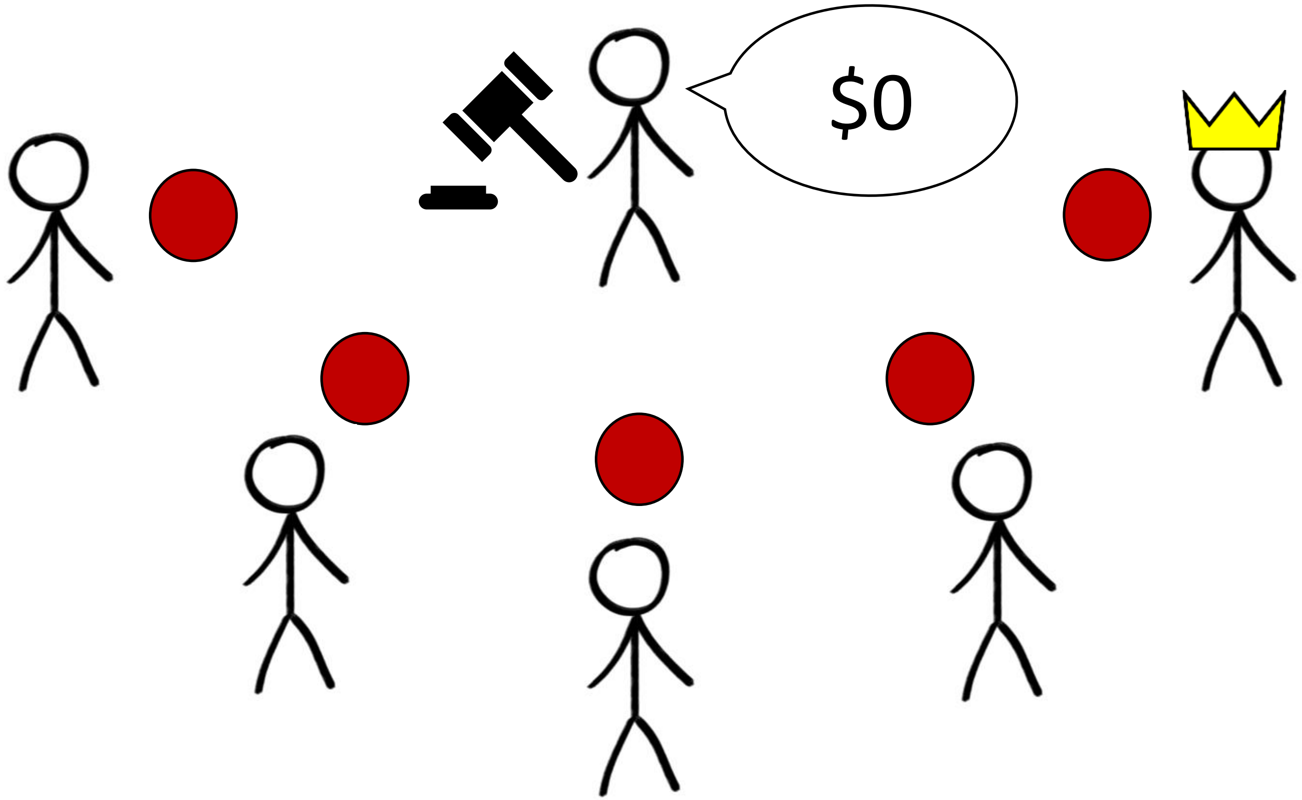
$$\text{Rev}[1^{\text{st}} \text{ price}] \leq \text{Rev}[2^{\text{nd}} \text{ price}] \leq \text{Rev}[\text{English}]$$

3. Hierarchy explained by a general Linkage Principle

English Auction Equilibrium

Recall: Equil. strategy factors in the anticipated learning from winning

- What does i learn if she wins the English auction?



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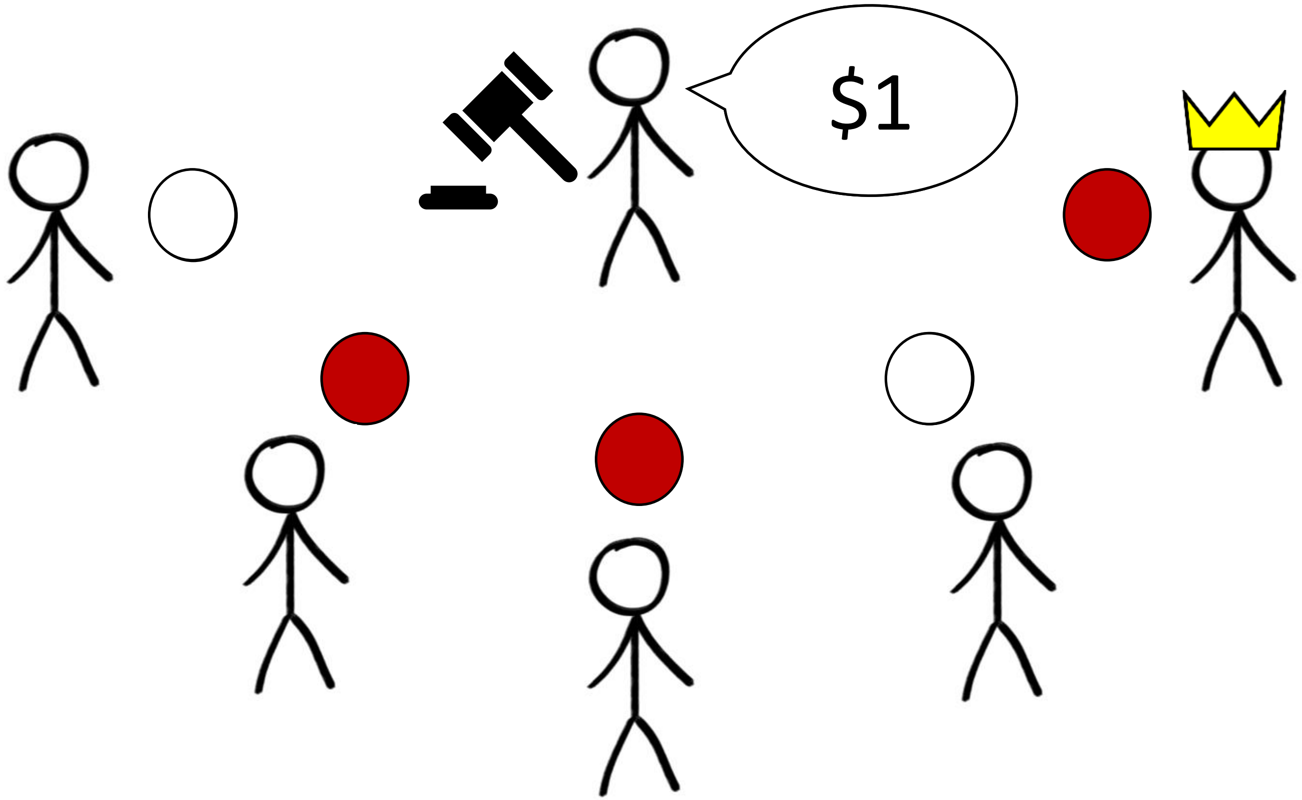

English Auction Equilibrium

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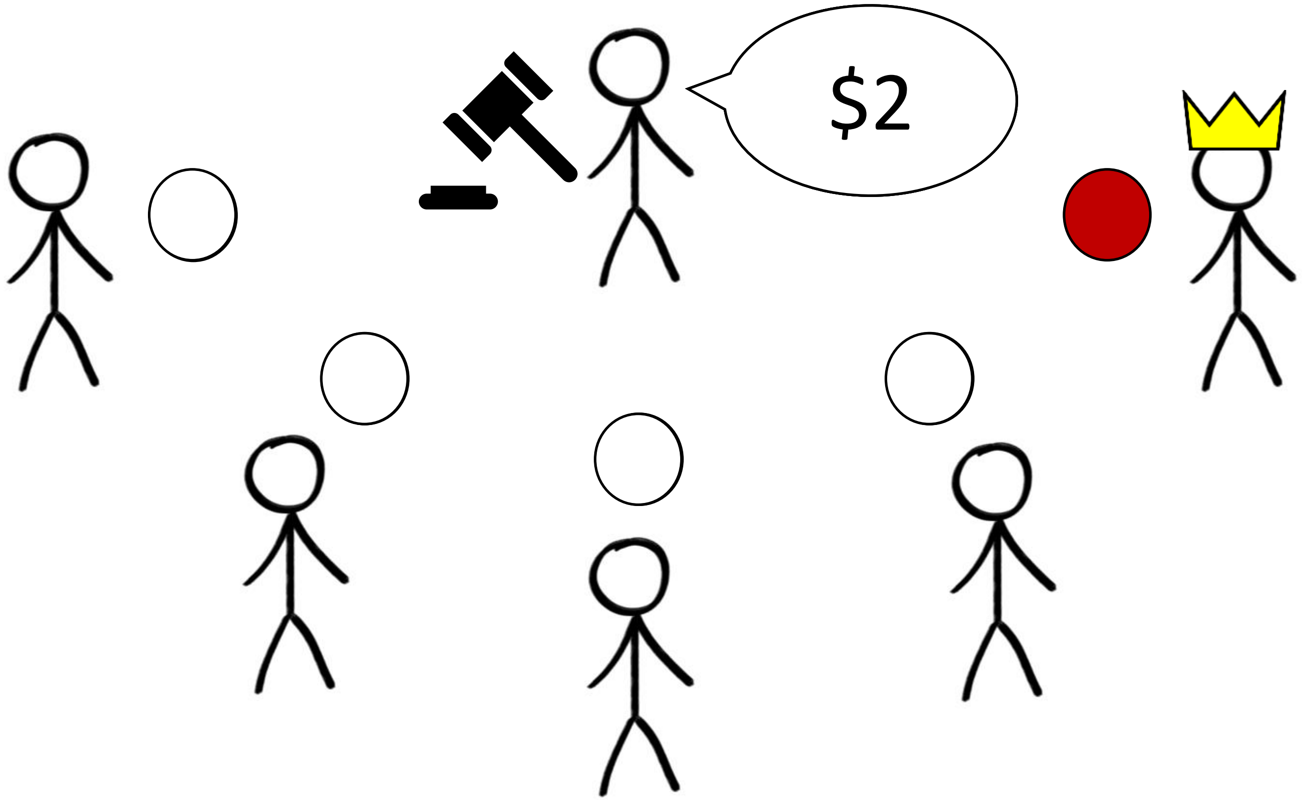


English Auction Equilibrium

<u>Recall</u>: Equil. strategy factors in the anticipated learning from winning

- What does i learn if she wins the English auction?
- **<u>Claim</u>**: All signals $\{X_j\}$!
  - Since all bidders use the same strategy to decide to drop out, when a bidder drops out her signal is revealed

<u>Equilibrium</u>: Bidding $u(X_i, \{X_j\}_{j \neq i})$ is impossible, so start with $u(x, \{x, \ldots, x\})$ and at each stage plug in the revealed X_j's

The Linkage Principal

- $A =$ auction in (symmetric, increasing) equilibrium
- $P^A(b, x) =$ equilibrium price if i wins with bid b and signal x
- $\underbrace{P^A(x)}_{\frac{\partial}{\partial x'}P^A(b, x') \mid_{(b,x')=(x,x)}} =$ statistical linkage of equilibrium price to i's signal x

  - For example, in 1st price auctions, $P^{first}(x) = 0$

Theorem [Informal]: For auctions A, B, if $P^A(x) \geq P^B(x)\ \forall x$ then

$\text{Rev}[A] \geq \text{Rev}[B]$

The Linkage Principal: Intuition

- **Theorem [Informal]**: For auctions A, B, if $\overbrace{P_2^A(x)}^{\text{Linkage of equil. price to winner's signal } x} \geq P_2^B(x)\ \forall x$ then

$$\text{Rev}[A] \geq \text{Rev}[B]$$

Intuition: Stronger linkage of the (public) price to the winner's private signal x lowers the information rent

- A.k.a. "publicity effect" [Milgrom'04]

Practical Take-Aways

- Popularity of English auctions
- “Honesty is the best policy”
  - If seller has information affiliated with that of bidders’, revealing it increases revenue
- Motivation for auction design

Auction Design & Robustness

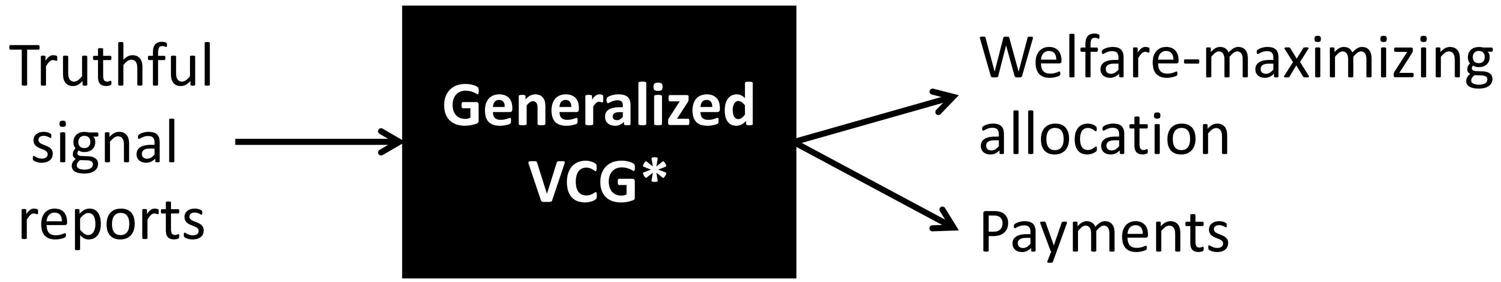
Optimal Auction [Cremer-McLean'88]

- **<u>We've seen</u>**: Correlation brings down information rents
- [Cremer-McLean'88] takes this to the extreme – down to zero!

<u>Main result</u>: A truthful optimal mechanism that extracts the full welfare as revenue

  - Generically, even the slightest degree of correlation among signals suffices

Optimal Mechanism: High-Level Idea



- Let $p_i(x_i)$ be the expected payoff (utility) of i from Gen-VCG given x_i
- Impose an additional "lottery" payment $\ell(x_{-i})$ on i such that:

$$\mathbb{E}_{x_{-i}|x_i}[\ell(x_{-i})] = p_i(x_i)$$

- Generically, such a lottery exists iff signals are correlated

Downside and Wilson's Doctrine

The optimal mechanism crucially depends on common knowledge of the joint signal distribution

"**I foresee the progress of game theory as depending on successive reductions in the base of common knowledge** required to conduct useful analyses of practical problems. Only by repeated weakening of common knowledge assumptions **will the theory approximate reality**"

[Wilson'87]

Summary So Far

Correlated and interdependent values $\neq$ IPV

- Bidders learn about their value from the auction (format matters!)
- Equilibrium bids anticipate this learning
- Correlation lowers information rents (possibly to zero!)

More after the break…

Recommended Further Reading

- Paul Milgrom, **Putting Auction Theory to Work**, Cambridge University Press, 2004
  - Chapter 5
- Vijay Krishna, **Auction Theory**, Elsevier, 2010
  - Chapters 6-10