Combinatorial Ski Rental and Online Bipartite Matching

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Our results:

• **optimal** (1-1/e)-competitive algorithms (against offline optimum) for combinatorial ski rental & online bipartite matching when costs / capacity constraints can be submodular

• no constant-factor algorithm exists when any part of our assumptions is relaxed
Combinatorial ski rental

- A business analyst's job involves 2 software products: Excel and Powerpoint.
- Tasks arrive over time, each of which may require either / both of the 2 products.
- In order to finish all tasks, analyst may "rent" or "buy" any combination of the products; purchased products cannot be returned.
- Discounts are available when renting or buying both products.

Cost of purchasing:
- Excel: $30
- Powerpoint: $30

Cost of renting:
- Excel: $30
- Powerpoint: $30

"Upgrading" is allowed: pay $20 for Powerpoint when Excel is already purchased.

<table>
<thead>
<tr>
<th>Task</th>
<th>Time</th>
<th>Product(s) Purchased</th>
<th>Cost so far</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>1st</td>
<td>1st</td>
<td>X P</td>
<td>$10</td>
<td>Rent Excel at cost $10, cost so far: $10</td>
</tr>
<tr>
<td>2nd</td>
<td>2nd</td>
<td>X P</td>
<td>$25</td>
<td>Rent both at cost $15, cost so far: $25</td>
</tr>
<tr>
<td>3rd</td>
<td>3rd</td>
<td>X P</td>
<td>$55</td>
<td>Buy Excel at cost $30, cost so far: $55</td>
</tr>
<tr>
<td>4th</td>
<td>4th</td>
<td>X P</td>
<td>$65</td>
<td>Rent Powerpoint at cost $10, cost so far: $65</td>
</tr>
<tr>
<td>5th</td>
<td>5th</td>
<td>X P</td>
<td>$85</td>
<td>Buy Powerpoint at (marginal) cost $20, cost so far: $85</td>
</tr>
</tbody>
</table>

Both products purchased, future cost = 0.
**Primal LP for ski rental**

- n products available for purchasing / renting
- f(S): cost of purchasing S
- g_t(S): cost of renting S at time t
- x(S) ≥ 0: probability of purchasing S
- y_t(S) ≥ 0: probability of renting S at time t
- minimize:
  \[ \sum_S x(S) f(S) + \sum_{1 \leq t \leq T} \sum_{S'} y_t(S') g_t(S') \]
- s.t. for all item i, time t:
  \[ \sum_{S: i \in S} x(S) + y_t(S) \geq 1 \]

**Dual LP for online matching**

- n offline vertices, one online vertex at each time t
- \( \lambda_t(i) \): fraction of online vertex arriving at time t matched to offline vertex i
- maximize:
  \[ \sum_{1 \leq t \leq T} \sum_{1 \leq i \leq n} \lambda_t(i) \]
  total (fractional) number of online vertices matched
- s.t. (offline capacity constraints) for every set of items S:
  \[ \sum_{1 \leq t \leq T} \sum_{i \in S} \lambda_t(i) \leq f(S) \]
  total load of all offline vertices in S cannot exceed \( f(S) \)
- s.t. (online supply constraints) for every set of items S, time t:
  \[ \sum_{i \in S} \lambda_t(i) \leq g_t(S) \]
  fraction of online vertex at time t matched to offline vertices in S cannot exceed \( g_t(S) \)

take-home message: online primal-dual analysis can go fully combinatorial
Sketch of algorithm

- minimize: $\sum_S x(S) f(S) + \sum_{1 \leq t \leq T} \sum_{S'} y_t(S') g_t(S')$
- s.t. for all $i, t$: $\sum_{S: i \in S} x(S) + y_t(S) \geq 1$

- in constraints: only marginal probabilities matter
- re-parametrize by marginal probabilities
- consider convex envelope = Lovasz extension

primal:

- minimize: $F(x_t(1), \ldots, x_t(n)) + \sum_{1 \leq t \leq T} G_t(y_t(1), \ldots y_t(n))$

new problem: global objective & constraints

Overview of lower bounds

- lemma: under certain conditions, local constraints satisfied imply global constraints satisfied & small ratio between primal / dual objectives
- problem boils down to local task: setting $\lambda_t(i)$ (which determine $x_t(i)$) to satisfy these certain conditions at each time $t$
- relatively easy when $x_t(i)$ are all different, highly nontrivial (and combinatorial) otherwise; latter (hard) case is not "zero-measure"

no constant-factor algorithm if:

- cost of purchasing is allowed to be XOS,
- cost of renting is allowed to be supermodular, or
- upgrading is not allowed (i.e., price of purchasing $S$ given $S'$ is $f(S)$ rather than $f(S | S')$)

in words: all our assumptions are necessary

symmetrization technique: prepare future clauses, choose realization by demanding right copy of item