Finiteness Assumptions in Game Theory

- Finiteness assumptions baked into many core Game Theory results. E.g.,
  - **Finite sets of agents.**
  - **Finite time horizon (e.g., finite-past horizon).**
  - **Finite data sets.**

- Usually play a real role in the analysis.

- Dropping finiteness usually translates into completely rewriting the finite-case proof w/added detail (no reduction) if not a completely new, even more elaborate, proof.

- Generalizing each result requires different specialized tools.
  - Existence of Nash equilibrium
    - Finite markets: Nash, 1951 (via Brouwer’s theorem)
    - Infinite markets: Peleg, 1968 (via Schauder’s theorem)
  - Existence of a stable matching
    - Finite markets: Gale and Shapley, 1962 (explicit algorithm)
    - Infinite markets: Fleiner, 2003 (via Tarski’s theorem)

Our paper: a principled, widely applicable, “user friendly” approach for lifting finite-model results as black boxes to infinite models.
Some Existence Results, and Challenges

Infinitely many agents

- Stable matching
  - Finite markets: Gale and Shapley, 1962
  - Infinite markets: Fleiner, 2003 (via Tarski’s theorem)
  - Strategyproofness in infinite markets? (Open: Jagadeesan, 2018)
    - Lone wolf thm doesn’t hold ⇒ new approach needed if in fact true.
    - Lift variant existence results? (Not DA based... proof via Tarski unlikely.)
  - Walrasian equilibrium in trading networks (substitutable prefs.)
    - Finite networks: Hatfield et al., JPE, 2013
    - Infinite networks? (Finite-network proof delicate, complex as it is.)

Infinite past horizon

- Dynamic stable matching with tenure
  - Finite start, infinite future-horizon (Pereyra, 2013, tweaked DA)
  - Infinite past-horizon?
    - No “men-optimal” stable matching... GS/DA-style proof unlikely.

Infinite observed data sets — Revealed-preference theory

- Rationalizability of finite demand datasets (GARP): Afriat, 1967
- Infinite demand datasets: Reny, ECMA, 2015 (new elaborate proof)
- Lift other rationalizability results? (e.g., McFadden and Richter, 1971)
  - Existing generalizations assume added structure or weaken rationalizability.
Propositional Logic 101: Logical Compactness

- Define a set of Boolean **variables**—“facts about our solution”
  - E.g., \( \{\text{matched}(m,w)\} \forall m, w \)
  - E.g., \( \{\text{price}(o,p)\} \forall o,p \quad / \quad \{\text{consumes}(a,o)\} \forall a,o \)

- The set of well-formed propositional **formulae** is defined inductively:
  - All atomic formulae—the variables defined above,
  - \( \neg \phi \) for every well-formed formula \( \phi \),
  - \( (\phi \lor \psi) \), \( (\phi \land \psi) \), \( (\phi \rightarrow \psi) \), and \( (\phi \leftrightarrow \psi) \) for every two well-formed formulae \( \phi \) and \( \psi \).

  E.g., \( \text{matched}(\tilde{m},\tilde{w}) \), \( \text{matched}(m,w') \lor \text{matched}(m',w) \),
  \( \neg(\text{matched}(m,w) \land \text{matched}(m,w')) \). (Each formula always finite!)

- A **model** is an assignment of (Boolean) truth values to the variables. The **truth** value of formulae in a model is defined inductively.

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**The Compactness Theorem for Propositional Logic**

A set of formulae is **satisfiable** (i.e., \( \exists \) model where all these formulae are TRUE) if and only if every finite subset thereof is satisfiable.
Instructive Example: Existence of a Stable Matching

- **Women** $W$, **men** $M$, each specifies $n$th choice spouse, $\forall n$.
- **Matching**: 1-to-1 between (some) women and (some) men.
- A matching is **blocked** by a pair $(m, w)$ if $m$ prefers $w$ to his match and $w$ prefers $m$ to her match. **Stable** if not blocked.
- *Not* a continuum model! Infinitely many nonnegligible players.
- **Gale and Shapley, 1962**: always exists in finite markets.

A compact, logical proof for infinite markets, by reduction

Formulae over the variables $\{\text{matched}(m, w)\}_{m \in M, w \in W}$:

- $\text{matched}(m, w) \rightarrow \neg \text{matched}(m, w')$ \quad $\forall m, w \neq w'$;
- $\text{matched}(m, w) \rightarrow \neg \text{matched}(m', w)$ \quad $\forall m \neq m', w$;
- $\neg \text{matched}(m, w)$ \quad $\forall m, w$ not mutually ranked;
- $\neg \text{matched}(m, w) \rightarrow ((\text{matched}(m, w_1) \lor \cdots \lor \text{matched}(m, w_l)) \lor \neg \text{matched}(m, w))$ \quad $\forall m, w$ where $w_1, \ldots, w_l \succ_m w$ and $m_1, \ldots, m_k \succ_w m$.

Finite formula! $\lor (\text{matched}(m_1, w) \lor \cdots \lor \text{matched}(m_k, w))$ $\forall m, w$.

Compactness $\Rightarrow$ enough to satisfy every finite formula set.

Every finite subset satisfiable by existence in finite markets. $\square$