Solution to Exchanges 10.3 Puzzle: Contingency Exigency

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First, a quick recap of the problem: the payout is a random variable X; we assume we know its distribution F, and in particular that we know $E_F[X]$. We are looking for a function $\omega(\cdot,\cdot)$ such that $E_F[\omega(t,X)] = r \cdot t$, where r is the non-contingency hourly rate. Initially, we impose that $\forall t, \omega(t,0) = 0$ and that ω is linear in t.

1. First, consider the problem with no minimum payment. The second condition implies that we can write $\omega(t,X) = t \cdot g(X)$ for some function g. There are infinitely many functions g which would work. The simplest one is to take $g(X) = \alpha \cdot X$. The formula $\omega(t,X) = \alpha \cdot X \cdot t$ means that the agent's hourly rate is a percentage of the total winnings (as opposed to the more classical scheme where the total payment is a percentage of the winnings, independently of time spent). All we have to do is find α. This is easy enough:

$$\begin{aligned} E_F[\omega(t,X)] &= r \cdot t \\ E_F[\alpha \cdot t \cdot X] &= r \cdot t \\ \alpha \cdot t \cdot E_F[X] &= r \cdot t \\ \alpha &= \frac{r}{E_F[X]} \end{aligned}$$

The solution is thus $\omega(t,X) = r \cdot X \cdot t / E_F[X]$. Note that we were given the condition $r \cdot t < E_F[X]$, which guarantees that $\omega(t,X) < X$, so you will not need to spend the entire payout to pay the agent.

2. Now consider the issue of a minimum hourly rate: $\omega(t,X) \geq m \cdot t$. We will look for a solution of the form $\omega(t,X) = t \cdot (m + \beta \cdot X)$: the hourly rate is composed of a fixed minimum, plus a percentage of the payout (obviously, this percentage will be less than the percentage α from part 1). As above, this simplifies to $m + \beta \cdot E_F[X] = r$, *i.e.* $\beta = (r - m)/E_F[X]$. The solution is thus $\omega(t,X) = m \cdot t + (r-m) \cdot t \cdot X / E_F[X]$.

In part 2 as in part 1, the hourly rate is greater than the non-contingency rate r iff the payout is greater than expected.

3. Finally, consider the issue of a minimum total payment: $\omega(t,X) \geq m$. In this case, it is fair to impose a minimum number of hours of work: $r \cdot t \geq m$ (if we do not add this condition, it is impossible to have $\omega(t,X) \geq m$ and $E_F[\omega(t,X)] = r \cdot t$ hold at the same time). We look for solutions of the form $\omega(t,X) = m + \gamma \cdot t \cdot X$. Taking expectations gives $r \cdot t = m + \gamma \cdot t \cdot E_F[X]$, *i.e.* $\gamma = (r - m/t)/E_F[X]$. We have $\gamma \geq 0$ because of the constraint we added. The solution is thus $\omega(t,X) = m + (r \cdot t - m) \cdot X / E_F[X]$.

An interpretation of this solution is as follows: the first m/r hours are paid at the non-contingency rate r. All hours above are paid as a percentage of the payout, with the same rate as in part 1. Note that with this solution, it is possible that $\omega(t, X) > X$ (for example if $X = 0$): you might incur a loss.