Informationally Robust Auction Design

Benjamin Brooks
UChicago

Songzi Du
UCSD

SIGecom Winter Meeting

February 25, 2021

Introduction

- Bob Wilson and Paul Milgrom:
  - Equilibrium bidding in standard auctions w/. **common** or **interdependent** values

- Next step: optimal auction design
  - Correlated values $\rightarrow$ Correlated signals $\rightarrow$ Crémer and McLean full surplus extraction

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- Next step: optimal auction design
  - Correlated values → Correlated signals → Crémer and McLean full surplus extraction
- Informationally robust auction design (à la Bergemann and Morris):
  - Ignore the correlation in signals
  - Focus on the correlation in values
  - Maximizes the worst-case revenue across all info structures

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- **Mechanism** $\mathcal{M} = (\{A_i\}, \{q_i\}, \{t_i\})$:
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- Bayes-Nash equilibrium given $(\mathcal{M}, \mathcal{I})$

Solution concept

- Seller's **maxmin mechanism** solves:

$$\sup_{\mathcal{M}} \inf_{\mathcal{I}} \inf_{\beta \in B(\mathcal{M},\mathcal{I})} \Pi(\mathcal{M}, \mathcal{I}, \beta), \tag{1}$$

  - $B(\mathcal{M}, \mathcal{I})$ is the set of equilibria,
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Strong Minimax Theorem

Suppose we restrict to finite mechanisms and info structures, then

$$\sup_{\mathcal{M}} \inf_{\mathcal{I}} \inf_{\beta \in B(\mathcal{M},\mathcal{I})} \Pi(\mathcal{M},\mathcal{I},\beta) = \inf_{\mathcal{I}} \sup_{\mathcal{M}} \sup_{\beta \in B(\mathcal{M},\mathcal{I})} \Pi(\mathcal{M},\mathcal{I},\beta).$$

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- Suppose bidders have common value: $v_1 = v_2 = \cdots = v_N$.
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- Maxmin mechanism is **proportional auction**: $\overline{A}_i = \mathbb{R}_+$ for each bidder i,

$$\overline{q}_i(a) = \frac{a_i}{\Sigma a} \cdot \overline{Q}(\Sigma a), \qquad \overline{t}_i(m) = \frac{a_i}{\Sigma a} \cdot \overline{T}(\Sigma a),$$

where $\Sigma a = \sum_{i=1}^N a_i$,

$$\overline{Q}(\Sigma a) = \begin{cases} \Sigma a / x & \Sigma a < x, \\ 1 & \Sigma a \geq x. \end{cases}$$

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- Price per unit is $\overline{T}(\Sigma a)/\overline{Q}(\Sigma a)$
- Reminiscent of a Tullock contest, and of Voucher auction in Russia in 1990s.

Common value setting

- Minmax info structure $\overline{\mathcal{I}}$ has $S_i = \mathbb{R}_+$, s_i i.i.d. exponentially distributed
- The interim expected value

$$\mathbb{E}[v \mid s] = \begin{cases} C \exp(\Sigma s) & \Sigma s < x, \\ H^{-1}(G_N(\Sigma s)) & \Sigma s \geq x, \end{cases}$$

where H is the CDF of v, and G_N is the CDF of Σs (Gamma distribution).

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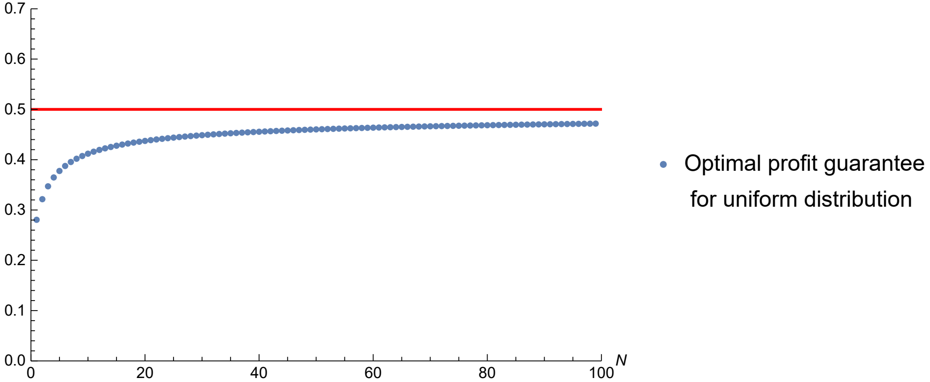
  where H is the CDF of v, and G_N is the CDF of Σs (Gamma distribution).
- Truth telling is an equilibrium of the proportional auction under $\overline{\mathcal{I}}$:
  - Proportional auction is the profit maximizing direct mechanism on $\overline{\mathcal{I}}$
  - $\overline{\mathcal{I}}$ is the profit minimizing correlated equilibrium on proportional auction.

Common value setting

Proposition

The optimal profit guarantee of the proportional auction converges to the full surplus $\mathbb{E}[v]$ *as* $N \to \infty$ *at the rate of* $\frac{1}{\sqrt{N}}$.



Why $\overline{A}_i = \overline{S}_i = \mathbb{R}_+$?

- Suppose $\overline{A}_i = \overline{S}_i = \mathbb{R}_+$.
- The Lagrangian for seller's **profit maximization** problem, fixing σ:

$$\begin{aligned}\mathcal{L} = &\sum_i \int_{\overline{A}\times V} t_i(a)\sigma(da, dv) \\ &+ \sum_i \int_{\overline{A}_i} \alpha_i(a_i) \int_{\overline{A}_{-i}\times V} \left[v_i \nabla_i q_i(a) - \nabla_i t_i(a)\right] \sigma(da, dv) \\ &+ \int_{\overline{A}} \gamma(a) \left[1 - \sum_i q_i(a)\right] da \\ &+ \int_V \lambda(v) \left[\int_{\overline{A}} \sigma(da, dv) - \mu(dv)\right]\end{aligned}$$

- Local IC:

$$\int_{\overline{A}_{-i}\times V} \left[v_i \frac{q_i(a_i, a_{-i}) - q_i(a_i - \epsilon, a_{-i})}{\epsilon} - \frac{t_i(a_i, a_{-i}) - t_i(a_i - \epsilon, a_{-i})}{\epsilon}\right] \sigma(da, dv) \geq 0$$

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- The **same Lagrangian** for both seller and Nature:
  - Seller choose direct mechanism on $\overline{A} = \mathbb{R}_+^N$ to maximize $\mathcal{L}$
  - Nature choose info structure on $\overline{S} = \mathbb{R}_+^N$ to minimize $\mathcal{L}$
  - A standard zero-sum game

Conclusion

- A novel auction format (proportional auction) to sell common value good
- Informationally robust auction design is especially appealing when values are highly correlated